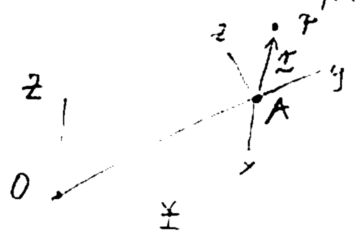
EM311M - Dynamics
Final Exam
Monday, May 16, 2011, 9 - noon, CPE 2.214
Version A

1. (5 points) Use the formula for the derivative of a vector-valued function of time in a rotating system of coordinates, to derive the formulas for the velocity and acceleration of a particle in the system.

$$\dot{f} = \dot{f}\big/_{rel} + \omega \times f \qquad OP = OA + \underset{r}{AP}$$



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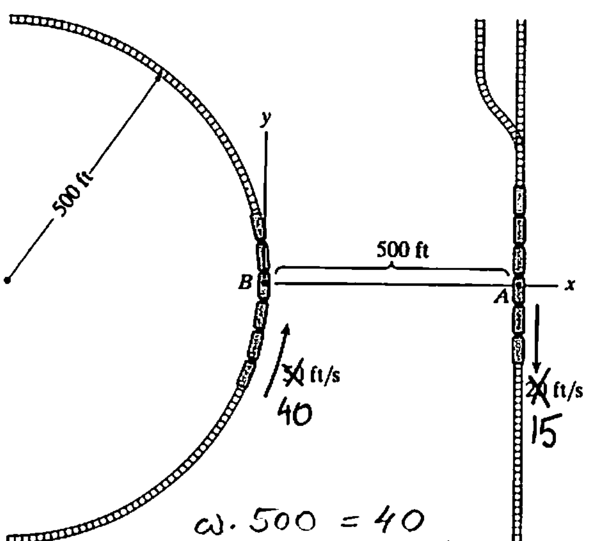
$$v_P = \dot{OP} = \dot{OA} + \dot{r}\big/_{rel} + \omega \times r$$
$$= v_A + v_{P\,rel} + \omega \times r$$
$$a_P = \ddot{OP} = \dot{v}_A + \dot{v}_{P\,rel}\big/_{rel} + \omega \times v_{P\,rel}$$
$$+ \underset{\alpha}{\dot{\omega}\big/_{rel}} \times r + \omega \times \dot{r}\big/_{rel} + \omega \times (\omega \times r)$$
$$= a_A + \alpha \times r + \omega \times (\omega \times r) + a_{P\,rel} + \underbrace{2\omega \times v_{P\,rel}}_{a_c}$$

$*$ $\dot{\omega} = \dot{\omega}\big/_{rel} + \overset{0}{\omega \times \omega} = \alpha$

2. The train on the circular track is traveling at a constant speed of 40 ft/s in the direction shown. The train on the straight track is traveling at 15 ft/s in the direction shown. Determine the relative velocity of passanger *A* observed by passenger *B* in a system of coordinates attached to train *B*. (5 points)



$$\omega \cdot 500 = 40$$
$$\omega = 0.08 \left[\frac{rad}{s}\right]$$

$$v_A = v_B + v_{A\,rel} + \omega \times BA$$
$$\therefore \quad v_{A\,rel} = v_A - v_B - \omega \times BA$$
$$= (0, -15, 0) - (0, 40, 0)$$
$$- \quad \begin{array}{l} \omega\,(0,\ 0,\ 0.08) \\ \times\ BA\,(500,\ 0,\ 0) \\ \hline (0,\ 40,\ 0) \end{array}$$
$$= (0, -95, 0) \quad \left[\frac{ft}{s}\right]$$

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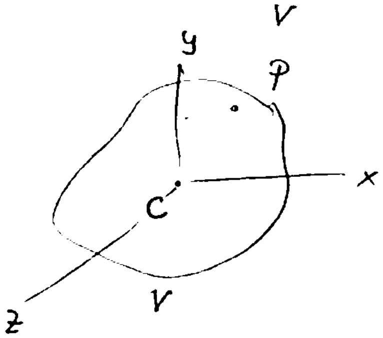
3. Derive the formula for the kinetic energy of a rigid body undergoing a planar motion. (5 points)

$$T = \frac{1}{2}\int_V \rho\, \underset{\sim}{v}^2 = \frac{1}{2}\int_V \rho\left[(v_{c_x} - \omega y)^2 + (v_{c_y} + \omega x)^2\right]$$



$$= \frac{1}{2}\int_V \rho\left[v_{c_x}^2 + v_{c_y}^2\right] - \omega v_{c_x} \underbrace{\int_V \rho y}_{0} + \omega v_{c_y} \underbrace{\int_V \rho x}_{0}$$

$$+ \frac{1}{2}\omega^2 \int_V \rho\left[x^2+y^2\right] = \underline{\frac{1}{2} m |\underset{\sim}{v}_c|^2 + \frac{1}{2} I_z \omega^2}$$

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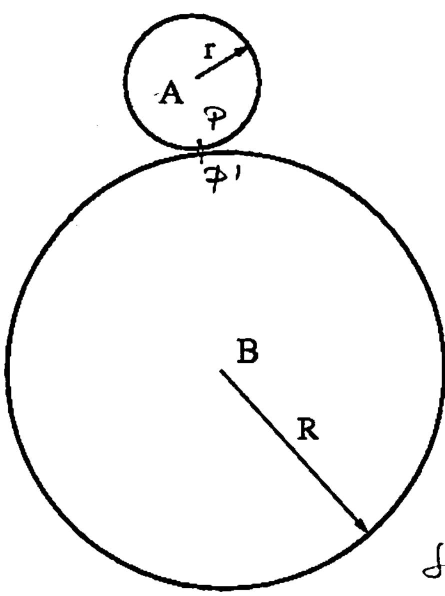
$$\underset{\sim}{v}_P = \underset{\sim}{v}_c + \underset{\sim}{\omega} \times \underset{\sim}{cP} = (v_{c_x}, v_{c_y}, 0) + \begin{array}{r} \times\,(0,\ 0,\ \omega) \\ (x,\ y,\ z) \\ \hline (-\omega y,\ \omega x,\ 0) \end{array}$$

4. Compute the kinetic energy of small disk A rotating *without slipping* along large disk B with angular velocity ω. (5 points)



$$T = \frac{1}{2} m v_A^2 + \frac{1}{2}\underbrace{\left(\frac{1}{2} m r^2\right)}_{I_A}\omega^2$$

Kinematics:

$$\underset{\sim}{v}_P = \underset{\sim}{v}_{P'} = \underset{\sim}{0} \quad (\text{no slipping})$$

$$\underset{\sim}{v}_A = \cancel{\underset{\sim}{v}_P}^{\,0} + \underset{\sim}{\omega} \times \underset{\sim}{PA} \qquad |\underset{\sim}{v}_A| = \omega r$$

So:

$$T = \frac{1}{2} m \omega^2 r^2 + \frac{1}{4} m r^2 \omega^2 = \frac{3}{4} m r^2 \omega^2$$

5. Derive the principle of angular impulse and momentum for a system of particles with respect to the center of mass of the system. (5 points)

Cl will start with the same principle but for a fixed point O.

$$\dot{H}_O = M_O$$

$$H_C = \sum_i CP_i \times m_i v_i = \sum_i (CO + OP_i) \times m_i v_i$$

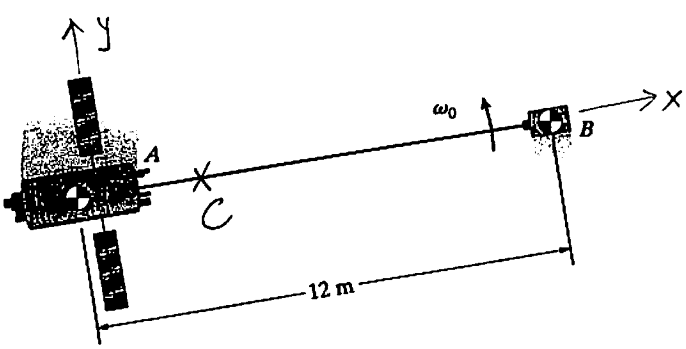
$$m_i v_i = CO \times \sum_i m_i v_i + H_O$$

$$\overrightarrow{P_i} \qquad \qquad \sum_i m_i v_i = M v_C$$

C, O

$$\dot{H}_C = \dot{CO} \times M v_C + CO \times M \dot{v}_C + \dot{H}_O = CO \times S' + M_O = M_C$$

6. Two gravity satellites (m_A =200 kg, m_B = 70 kg are tethered by a cable and rotating as a rigid body with respect to the center of mass of the combined system with angular velocity ω = 3 revolutions per minute. Compute the angular momentum of the system with respect to its center of mass. (5 points)



$$x_c = \frac{70 \cdot 12}{270} = 3.11\ [m]$$

$$I_C = 200 \cdot 3.11^2 + 70 \cdot (12 - 3.11)^2$$
$$= 1935.8 + 5530.8$$
$$= 7466.7\ [kg\, m^2]$$

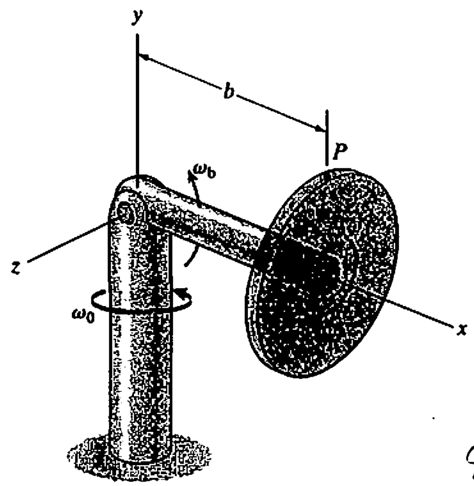
$$H_c = I_c \omega = 2345.7\ [kg \tfrac{m}{s}\ m]$$

(5)

$$\omega = 3 \frac{rev}{min} = \frac{3 \cdot 2\pi}{60} = 0.314\ [\tfrac{rad}{s}]$$

(5)

7. Compute the angular velocity of the disk with respect to a stationary frame in the system of coordinates attached to the bar. (5 points)



angular velocity of the shaft
wrt the stationary frame
$\underset{\sim}{\omega}_1 = (0, \omega_0, 0)$

angular velocity of the bar
wrt the shaft $\underset{\sim}{\omega}_2 = (0, 0, \omega_b)$

angular velocity of the disk
wrt the bar
$\underset{\sim}{\omega}_3 = (\omega_d, 0, 0)$

$\underset{\sim}{\omega} = \underset{\sim}{\omega}_1 + \underset{\sim}{\omega}_2 + \underset{\sim}{\omega}_3 = \underline{(\omega_d, \omega_0, \omega_b)}$

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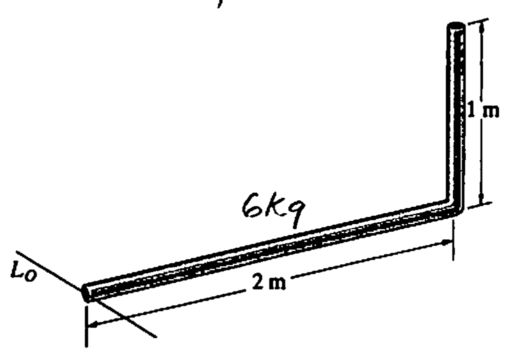
8. Use known formulas (or integrate if you forgot them) and Parallel Axes Theorem to compute the moment of inertia of the homogeneous bar of mass $m = 9$ kg shown below. (5 points)
with respect axis L_o



$$I_{L_o} = \frac{1}{12} 6\, 2^2 + 6 \cdot 1^2 + \frac{1}{12} 3\, 1^2 + 3 \cdot (2^2 + 0.5^2)$$

$$= 2 + 6 + 0.25 + 12.75$$

$$= 21 \ [kg\, m^2]$$

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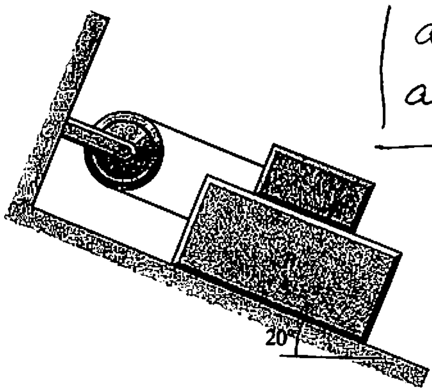
9. If A weighs 20 lb, B weighs 90 lb, and the coefficient of kinetic friction between all surfaces is $\mu_k = 0.12$, what is the tension in the cord as B slides down the inclined surface? (25 points)

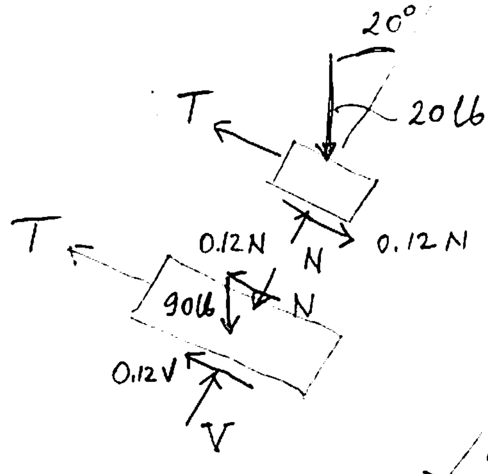


$a_{Bx} = a$

$a_{Ax} = -a$

a

a



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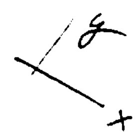

Block A:

$\sum F_y = 0 : \quad N - 20\cos 20° = 0 \quad \Rightarrow N = 18.79$ [lb]

$m\, a_{Ax} = \sum F_x : \quad \frac{20}{32.2}(-a) = 20 \sin 20° + 0.12 \cdot 18.79 - T$

$\therefore \quad T = 9.095 + 0.62a$

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Block B:

$\sum F_y = 0 \quad : \quad V - 90 \cos 20° - 18.79 = 0 \quad \Rightarrow V = 103.36$ [lb]

$m\, a_{Bx} = \sum F_x : \quad \frac{90}{32.2} a = 90 \sin 20° - 0.12 \cdot 103.36$

$- 0.12 \cdot 18.79 - T$

$\therefore \quad T = 16.124 - 2.795a$

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Comparing.

$9.085 + 0.62a = 16.124 - 2.795a$

$3.415 a = 7.029$

$a = 2.058$ [ft/s²]

$\therefore T = \underline{10.37}$ [lb]

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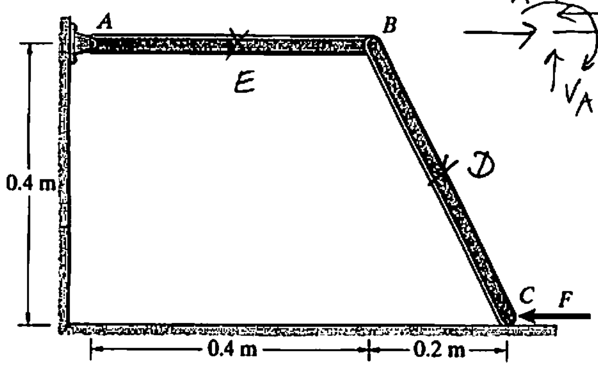
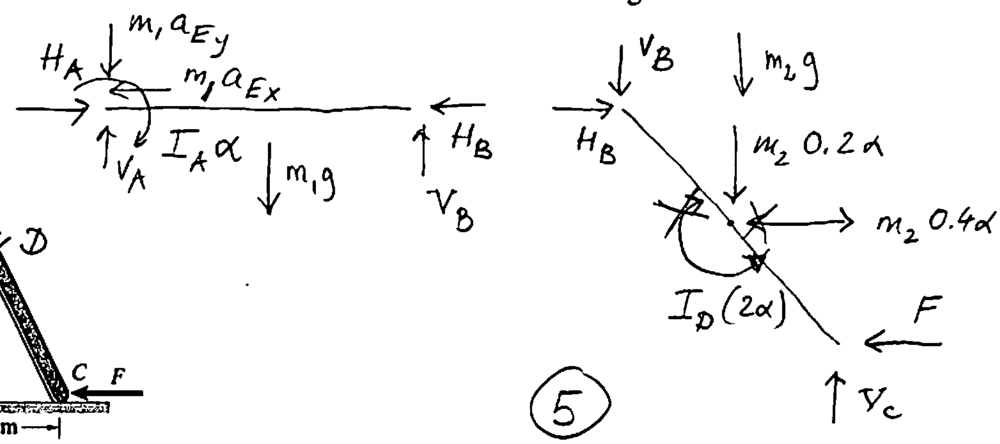
10. The masses of the slender bars AB and BC are 8 kg and 10 kg, respectively. The system starts from rest and the horizontal force is $F = 140\text{N}$. The horizontal surface is smooth. Determine the angular accelerations of the bars. (25 points)





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Kinematics :

starts from rest $\Rightarrow$ all velocities $= 0$

$$\underset{\sim}{a}_B = \cancel{\underset{\sim}{a}_A}^{0} + \underset{\sim}{\alpha}_{AB} \times \underset{\sim}{AB} = \quad \begin{array}{r} \underset{\sim}{\alpha}_{AB}\ (0,\ 0,\ \alpha_{AB}) \\ \times\quad \underset{\sim}{AB}\ (0.4,\ 0,\ 0) \\ \hline (0,\ 0.4\alpha_{AB},\ 0) \end{array}$$

$$\underset{\sim}{a}_C = \underset{\sim}{a}_C + \underset{\sim}{\alpha}_{CB} \times \underset{\sim}{CB} = (a_{Cx}, 0, 0) + \quad \begin{array}{r} \underset{\sim}{\alpha}_{CB}\ (0,\ 0,\ \alpha_{CB}) \\ \times\quad \underset{\sim}{CB}\ (-0.2,\ 0.4,\ 0) \\ \hline (-0.4\alpha_{CB},\ -0.2\alpha_{CB},\ 0) \end{array}$$

$$a_{Cx} - 0.4\alpha_{CB} = 0$$

$$0.4\alpha_{AB} = -0.2\alpha_{CB}$$

Set $\alpha_{AB} =: \alpha$, $\alpha_{CB} = -2\alpha$, $a_{Cx} = -0.8\alpha$

Acceleration of center of mass D of bar CB

$$\underset{\sim}{a}_D = \underset{\sim}{a}_C + \underset{\sim}{\alpha}_{CB} \times \underset{\sim}{CD} = (-0.8\alpha, 0, 0) + \quad \begin{array}{r} \alpha_{CB}\ (0,\ 0,\ -2\alpha) \\ \times\quad \underset{\sim}{CD}\ (-0.1,\ 0.2,\ 0) \\ \hline (0.4\alpha,\ 0.2\alpha,\ 0) \end{array}$$

$$= (-0.4\alpha,\ 0.2\alpha,\ 0)$$

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Problem 10 - continued

$$\ell_{BC}^2 = 0.2^2 + 0.4^2 = 0.2$$

Bar CB : $M_B = 0$

$$V_c \cdot 0.2 - F\,0.4 + 0.4 m_2 \alpha \cdot 0.2 - 0.2 m_2 \alpha \cdot 0.1 + \tfrac{1}{12} m_2\, 0.2 (2\alpha) - m_2 g\, 0.1 = 0$$

$$0.2 V_c = 0.4F + \alpha m_2 (-0.08 + 0.02 - 0.0333) + 0.98 m_2$$

$$V_c = 2F - 0.467\,\alpha m_2 + 4.91\, m_2$$

Bars combined ; $M_A = 0$

$$V_c \cdot 0.6 - F \cdot 0.4 + 0.4 m_2 \alpha \cdot 0.2 - 0.2 m_2 \alpha\, 0.5 + \tfrac{1}{12} m_2\, 0.2 (2\alpha)$$
$$- m_2 g \cdot 0.5 - m_1 g \cdot 0.2 - \underbrace{\tfrac{1}{3} m_1\, 0.4^2}_{I_A} \alpha = 0$$

$$0.6 V_c = 0.4F + \alpha m_2 (-0.08 + 0.1 - 0.0333) + 0.0533\, m_1 \alpha + 4.91 m_2 + 1.96 m_1$$

$$V_c = 0.667F - 0.0222\,\alpha m_2 + 0.0888\,\alpha m_1 + 8.18 m_2 + 3.27 m_1$$

Comparing V_c's ;

$$2F - 0.467\,\alpha m_2 + 4.91 m_2 = 0.667F - 0.0222\,\alpha m_2 + 0.088\,\alpha m_1 + 8.18 m_2 + 3.27 m_1$$

$$\alpha [0.444 m_2 + 0.0888 m_1] = 1.333F - 3.27 m_2 - 3.27 m_1$$

$$\alpha = \frac{127.81}{5.151} = \underline{24.8} \left[\frac{m}{s^2}\right]$$

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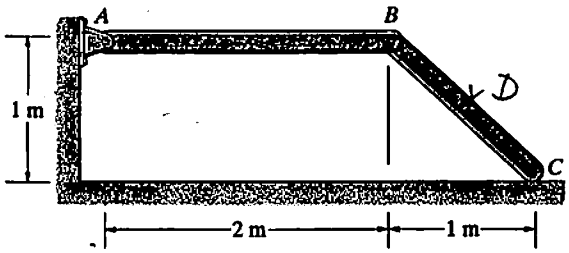
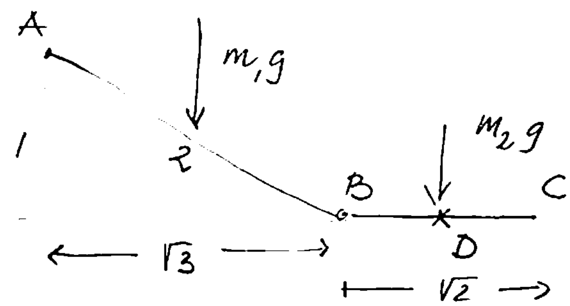
11. The masses of bars AB and BC are 4 kg and 2 kg, respectively. If the system is released from rest in the position shown, what are the angular velocities of the bars at the instant before the joint B hits the smooth floor? (25 points)





Kinematics at final position:

$$v_B = \cancel{v_A}^{\,0} + \omega_{AB} \times AB = \begin{array}{l} \omega_{AB}\ (0,\ 0,\ \omega_{AB}) \\ \times\ AB\ (\sqrt{3},\ -1,\ 0) \\ \hline (\omega_{AB},\ \sqrt{3}\,\omega_{AB},\ 0) \end{array}$$

Set $\omega_{AB} = \omega$

$$v_C = v_B + \omega_{BC} \times BC = (\omega, \sqrt{3}\omega, 0) + \begin{array}{l} \omega_{BC}\ (0,\ 0,\ \omega_{BC}) \\ \times\ BC\ (\sqrt{2},\ 0,\ 0) \\ \hline (0,\ \sqrt{2}\,\omega_{BC},\ 0) \end{array}$$

$$v_{Cy} = 0 \Rightarrow \sqrt{3}\,\omega + \sqrt{2}\,\omega_{BC} = 0$$

$$\therefore\ \omega_{BC} = -\sqrt{\tfrac{3}{2}}\,\omega$$

Velocity of center of mass D:

$$v_D = v_C + \omega_{CD} \times CD = (\omega, 0, 0) + \begin{array}{l} \omega_{BC}\ (0,\ 0,\ -\sqrt{2}\,\omega) \\ \times\ CD\ (-\tfrac{\sqrt{2}}{2},\ 0,\ 0) \\ \hline (0,\ \omega,\ 0) \end{array}$$

$$= (\omega, \omega, 0)$$

Kinetic energy of the system at the final moment:

$$T = \frac{1}{2}\underbrace{\frac{1}{3}m_1 \cdot 4}_{I_A}\,\omega^2 + \frac{1}{2}\underbrace{\frac{1}{12}m_2 \cdot \cancel{2}}_{I_D}\,\frac{3}{\cancel{2}}\omega^2 + m_2\,\omega^2$$

$$= (0.667\, m_1 + 1.125\, m_2)\,\omega^2 = 4.918\,\omega^2$$

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Problem 11 – continued

Principle of Work and Energy.

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$$T_1 + U_{12} = T_2$$

$$0 + 4 \cdot 9.81 \cdot 0.5 + 2 \cdot 9.81 \cdot 0.5 = 4.918\,\omega^2$$

$$\omega_{AB} = \omega = -2.45 \left[\frac{rad}{s}\right]$$

$$\omega_{BC} = -\sqrt{\tfrac{3}{2}}\,\omega = 3.00 \left[\frac{rad}{s}\right]$$

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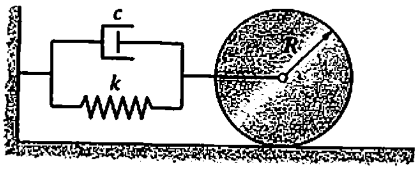
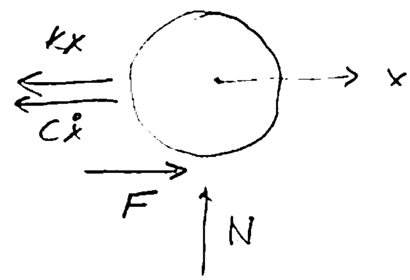
12. The homogeneous disk weighs 90 lb and its radius is $R = 1$ft. It rolls on the plane surface. The spring constant is $k = 100$ lb/ft and the damping constant is $c = 2$ lb-s/ft. Determine the frequency of small vibrations of the disk relative to its equulibrium position. (25 points)





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$$m\ddot{x} = F - kx - c\dot{x}$$

$$\underbrace{\tfrac{1}{2} m R^2}_{I} \alpha = F \cdot R \quad \Rightarrow \quad F = \tfrac{1}{2} m R \alpha$$
$$= -\tfrac{1}{2} m \ddot{x}$$

Kinematics : $-R\theta = x$

$\therefore \quad -R\alpha = \ddot{x}$

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$$m\ddot{x} = -\tfrac{1}{2} m\ddot{x} - kx - c\dot{x}$$

$$\tfrac{3}{2} m \ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \underbrace{\frac{2c}{3m}}_{2\gamma} \dot{x} + \underbrace{\frac{2k}{3m}}_{\omega_0^2} x = 0$$

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$$\omega^2 = \omega_0^2 - \gamma^2 = \frac{2 \cdot 100}{3 \cdot 2.795} - \left(\frac{2}{3 \cdot 2.795}\right)^2$$

$$m = \frac{90}{32.2} = 2.795 \text{ [slug]}$$

$$= 23.85 - 0.0568 = 23.8$$

$$\omega = 4.88 \left[\frac{\text{rad}}{s}\right]$$