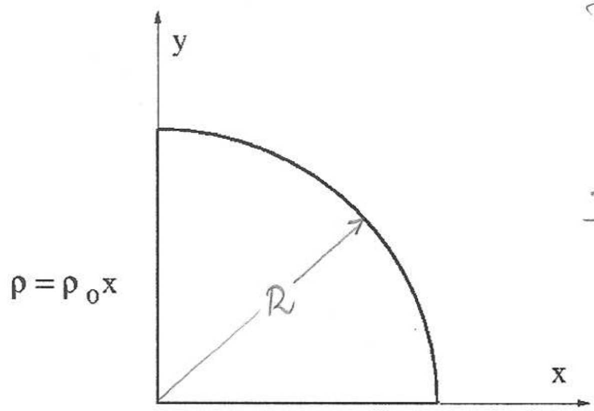
EM311M - Dynamics
Exam 3
Wednesday, May 4, 2011, 6:00-9:00 p.m., ECJ 1.202
Version A

1. Use polar coordinates to compute moment of inertia I_x for a *non-homogeneous* quadrant of a circle shown below. The density $\rho(x,y) = \rho_0 x$ where ρ_0 is a constant. (5 points)



Parametrization:

$$\begin{cases} x = r\cos\theta & 0 < r < R \\ y = r\sin\theta & 0 < \theta < \frac{\pi}{2} \end{cases}$$

$$I_x = \int_0^R \int_0^{\frac{\pi}{2}} \underbrace{\rho_0\, r\cos\theta}_{\rho}\ \underbrace{(r\sin\theta)^2}_{y^2}\ r\,dr\,d\theta$$

$$= \rho_0 \int_0^R r^4\,dr \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta\,d\theta$$

$$= \rho_0 \frac{r^5}{5}\Big|_0^R \ \frac{1}{3}\sin^3\theta\Big|_0^{\frac{\pi}{2}} = \rho_0 \frac{R^5}{5}\cdot\frac{1}{3} = \boxed{\frac{1}{15}\rho_0 R^5}$$

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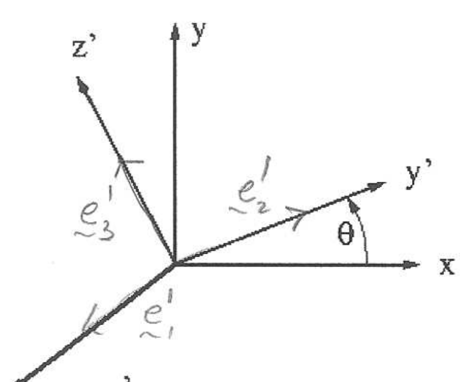
2. Define the transformation matrix from system xyz to system $x'y'z'$ and compute it for the systems shown below. (5 points)



$$\alpha_{ij} = \underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j$$

$\underset{\sim}{e}_i'$ – unit vectors of $x'y'z'$

$\underset{\sim}{e}_j$ – — " — of xyz

$$\underset{\sim}{\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}$$

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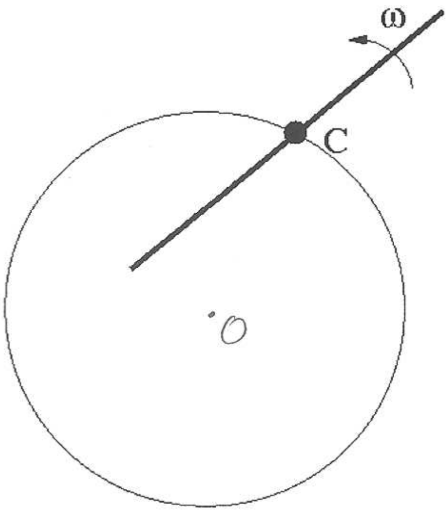
3. Center of mass of a slender bar with mass m and length l is rotating in circle of radius $l/2$ with an angular velocity ω. At the same time, the bar is rotating with respect to its center of mass with the same angular velocity ω. Compute the angular momentum of the bar with respect to the center of the circle. (5 points)



$$H_O = m \underbrace{\omega \frac{l}{2}}_{v} \cdot \frac{l}{2} + \underbrace{\frac{1}{12} m l^2}_{I_c} \omega$$

$$= \left(\frac{1}{12} + \frac{1}{4}\right) m l^2 \omega = \boxed{\frac{ml^2}{3}\omega}$$

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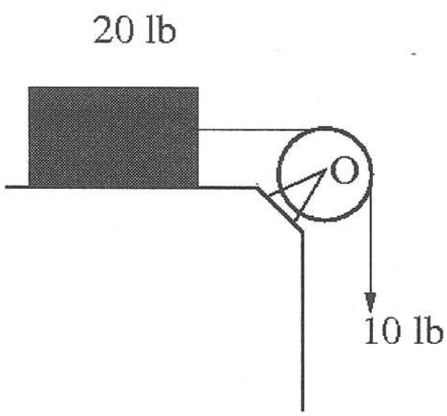
4. The system depicted below starts from rest and the surface is smooth. Consider two cases: case 1: inertia of the pulley is neglible, $I_0 \approx 0$, and case 2: inertia of the pulley is significant and must be taken into account, $I_0 > 0$. In which case will the acceleration of the block be smaller ? Explain. (5 points)



The second. Global inertia of the system is bigger.

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5. Derive the equation of angular motion for a rigid body undergoing an arbitrary motion, in the body-fitted system of coordinates. You may start from the principle of angular impulse and momentum. (5 points)

$$(I_c \omega)^{\circ} = M_c$$

$$\boxed{\underbrace{I_c \dot{\omega}/_{rel}}_{=\alpha} + \underbrace{\Omega}_{=\omega} \times (I_c \omega) = M_c}$$

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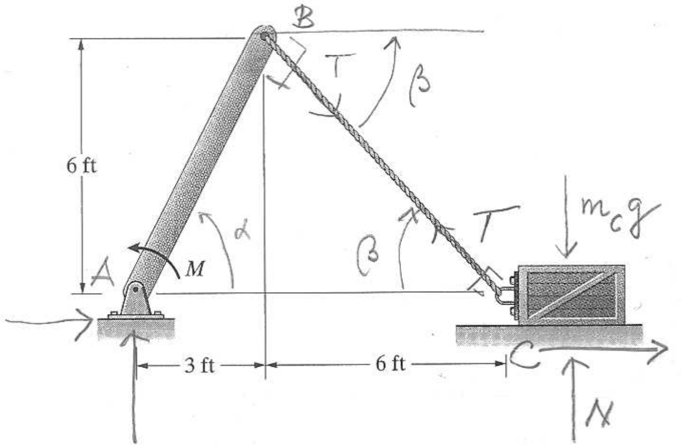
$$\alpha = \dot{\omega} = \dot{\omega}/_{rel} + \underbrace{\cancel{\omega \times \omega}}_{0} = \dot{\omega}/_{rel}$$

6. The slender bar weighs 50 lb, and the cart weighs 90 lb. At the instant shown, the velocity of the crate is zero and it has an acceleration of 15 ft/s^2 toward the left. The coefficient of kinetic friction between the horizontal surface and the crate is $\mu_k = 0.3$. Determine the couple M. (25 points)



The crate:

$$\Sigma F_y = 0 \Rightarrow N = m_c g - T\sin\beta$$

$$m_c a_{c_x} = \Sigma F_x \Rightarrow$$

$$-T\cos\beta + \mu_K N = m_c a_x$$

$$\Rightarrow T(\cos\beta + \mu_K \sin\beta) = \mu_K W_c - m_c a_x$$

$$\Leftrightarrow \boxed{T = \frac{\mu_K W_c - m_c a_x}{\cos\beta + \mu_K \sin\beta}}$$

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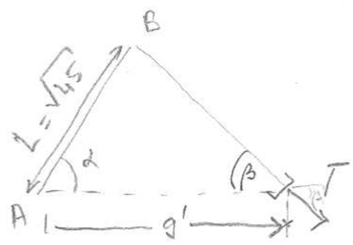
The bar

$$I_A \alpha_{bar} = M_A = -T\sin\beta \cdot 9 + M - W_b \frac{3}{2}$$



So:

$$M = I_A \alpha_{bar} + T\sin\beta \cdot 9 + \frac{3W_b}{2}$$

$$\boxed{= I_A \alpha_{bar} + \frac{\mu_K W_c - m_c a_x}{(1 + \mu_K \underbrace{\tan\beta}_{=1})} \underbrace{\tan\beta}_{=1} \cdot 9 + \frac{3W_b}{2}}$$

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Kinematics:

$$\underset{\sim}{v}_{crate} = \underset{\sim}{0} \Rightarrow \omega^{bar} = \omega^{rope} = 0$$

$$\underset{\sim}{a}_B = \cancel{\underset{\sim}{a}_A}^{\,0} + \underset{\sim}{\alpha}_{bar}(0, 0, \alpha_b) \times \underset{\sim}{AB}(3, 6, 0) = (-6\alpha_b, 3\alpha_b, 0)$$

$$\underset{\sim}{a}_B = \underset{\sim}{a}_c + \underset{\sim}{\alpha}_{rope}(0, 0, \alpha_r) \times \underset{\sim}{CB}(-6, 6, 0) = (a_{cx} - 6\alpha_r, -6\alpha_r, 0)$$

where $\underset{\sim}{\alpha}_{rope} \times \underset{\sim}{CB} = (-6\alpha_r, -6\alpha_r, 0)$

Comparing

$$3\alpha_b = -6\alpha_r \Rightarrow \boxed{\alpha_r = -\frac{1}{2}\alpha_b}$$

$$-6\alpha_b = a_{cx} - 6\alpha_r = a_{cx} + 3\alpha_b \Rightarrow \boxed{\alpha_B = -\frac{1}{9} a_{cx} = -\frac{1}{9} a_x}$$

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Final result:

$$\boxed{M = \frac{1}{3} m_{bar} \underbrace{45}_{L^2} \underbrace{\left(-\frac{1}{9} a_x\right)}_{\alpha_b} + 9 \frac{\mu_K W_c - m_c a_x}{1 + \mu_K} + \frac{3W_b}{2}} = \boxed{591 \text{ [ft} \cdot \text{lb]}}$$

$\boxed{T = 75 \text{ lb}}$

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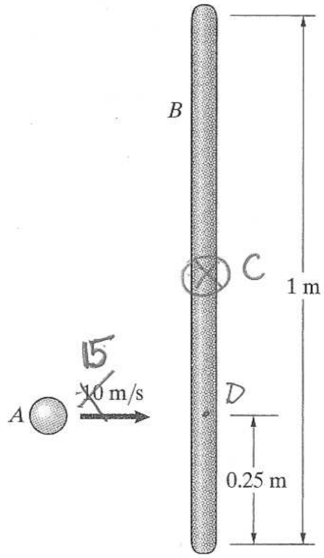
7. The 3-kg sphere A is moving toward the right at 15 m/s when it strikes the unconstrained 5-kg slender bar B. The coefficient of restitution is $e = 0.5$. What is the angular velocity of the bar after the impact? (25 points)



D – a fixed point (not on the bar...) coinciding with the point of impact.

Conservation of linear momentum in x direction for the system

$$m_A v_{Ax}^b + 0 = m_A v_{Ax}^a + m_B v_{Cx}^a \qquad (1)$$

5p

Conservation of angular momentum wrt fixed point D for the system

$$(2) \qquad 0 = 0 + (-\cancel{m_B} v_{Cx}^a)\cdot 0.25 + \frac{1}{12}\cancel{m_B}\cdot 1^2 \omega_B^a$$

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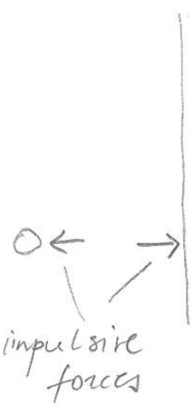
Conservation of linear momentum in y direction for the bar $\Rightarrow v_{Cy}^a = 0$



Kinematics: D' – point on the bar that coincides with D

$$\underset{\sim}{v}_{D'} = \underset{\sim}{v}_C + \underset{\sim}{\omega}_B \times \underset{\sim}{CD'}$$

$$= (v_{Cx}^a, 0, 0) + (0, 0, \omega_B^a) \times (0, -0.25, 0) \qquad [(0.25\,\omega_B^a, 0, 0)]$$

$$(3) \qquad = (v_{Cx}^a + 0.25\,\omega_B^a, 0, 0)$$

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Coefficient of restitution:

$$e = \frac{v_{D'x}^a - v_{Ax}^a}{v_{Ax}^b - \cancel{v_{D'x}^b}\,0} \Rightarrow v_{D'x}^a = v_{Ax}^a + e\, v_{Ax}^b \qquad (4)$$

5p

∴

(2) $\Rightarrow$ $\omega_B^a = 12 \cdot 0.25\, v_{Cx}^a = 3 v_{Cx}^a$

(3) $\Rightarrow$ $v_{D'x}^a = v_{Cx}^a + 0.25 \cdot 3 v_{Cx}^a = 1.75\, v_{Cx}^a$

(1) + (4) $\Rightarrow$

$$m_A v_{Ax}^a + m_B v_{Cx}^a = m_A v_{Ax}^b$$

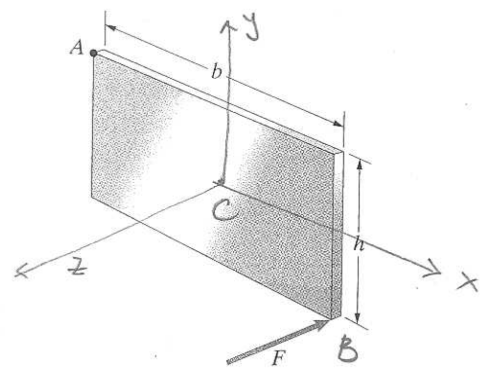
$$- v_{Ax}^a + 1.75\, v_{Cx}^a = e\, v_{Ax}^b$$

$$\therefore \quad v_{Cx}^a = \frac{\begin{vmatrix} m_A & m_A v_{Ax}^b \\ -1 & e\, v_{Ax}^b \end{vmatrix}}{\begin{vmatrix} m_A & m_B \\ -1 & 1.75 \end{vmatrix}} = \frac{m_A v_{Ax}^b (e+1)}{1.75 m_A + m_B} = \frac{67.5}{10.25} = 6.59 \left[\frac{m}{s}\right]$$

$$\omega_B^a = 3 v_{Cx}^a = \underline{19.76 \left[\frac{rad}{s}\right]}$$

5p

8. The dimensions of the 30-kg thin plate are $h = 0.5$m and $b = 0.7$m. The plate is stationary relative to an inertial reference frame when the force $F = 20$N is applied in the direction perpendicular to the plate. *No other forces or couples act on the plate.* What is the acceleration of point A at the instant when the force is applied ? (25 points)



$$I_x = \frac{1}{12} m h^2 = 0.625\ [kg\,m^2]$$

$$I_y = \frac{1}{12} m b^2 = 1.225\ [kg\,m^2]$$

$$I_z = I_x + I_y = 1.85\ [kg\,m^2]$$

$$I_{xy} = I_{yz} = I_{xz} = 0$$

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Translational motion eqn

$$m\,\underset{\sim}{a}_c = \underset{\sim}{F} \Rightarrow a_{cx} = a_{cy} = 0$$

$$a_{cz} = -0.667\ \left[\frac{m}{s^2}\right]$$

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Rotational motion eqn

$$\underset{\sim}{I}_c \underset{\sim}{\alpha} + \cancel{\underset{\sim}{\omega}}^{\,0} \times (\underset{\sim}{I}_c \underset{\sim}{\omega}) = \underset{\sim}{M}_c = \begin{array}{rl} & \underset{\sim}{CB}\,(0.35,\ -0.25,\ 0) \\ \times & \underset{\sim}{F}\,(\ \ 0,\ \ 0,\ -20) \\ \hline & (\ 5,\ \ 7,\ \ 0) \end{array}$$

$$0.625\,\alpha_x = 5 \Rightarrow \alpha_x = 8\ \left[\frac{rad}{s^2}\right]$$

$$1.225\,\alpha_y = 7 \Rightarrow \alpha_y = 5.71\ \left[\frac{rad}{s^2}\right]$$

$$1.85\,\alpha_z = 0 \Rightarrow \alpha_z = 0$$

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Kinematics

$$\underset{\sim}{a}_A = \underset{\sim}{a}_c + \cancel{\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{CA})}^{\,0} + \underset{\sim}{\alpha} \times \underset{\sim}{CA}$$

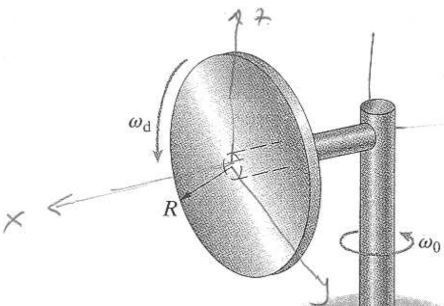
$$= (0,\ 0,\ -0.667) + \begin{array}{rl} & (8,\ \ 5.71,\ \ 0) \\ \times & (-0.35,\ 0.25,\ 0) \\ \hline & (0,\ \ 0,\ \ 4) \end{array}$$

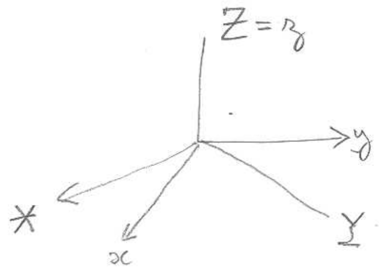
$$= (0,\ 0,\ 3.33)\ \left[\frac{m}{s^2}\right]$$

(5)

9 (bonus). The radius of the 5-kg disk is $R = 0.3$m. The disk is pinned to the horizontal shaft and rotates with a constant angular velocity $\omega_d = 3$ rad/s. Determine the magnitude of the couple exerted on the disk by the horizontal shaft.

(25 points)





XYZ − stationary

xyz − attached to the shaft (lower case)

$\underset{\sim}{\omega}_1$ = angular velocity of xyz (shaft) wrt XYZ = $(0, 0, \omega_0)$ in XYZ xyz

$\underset{\sim}{\omega}_2$ = angular velocity of disk wrt xyz = $(\omega_d, 0, 0)$ in xyz

We compute in <u>xyz</u> $\underset{\sim}{\Omega} = \underset{\sim}{\omega}_1 = (0, 0, \omega_0)$

ang. velocity of disk wrt XYZ $\underset{\sim}{\omega} = \underset{\sim}{\omega}_1 + \underset{\sim}{\omega}_2 = (\omega_d, 0, \omega_0)$ in xyz

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Angular Motion eqn

$$\underset{\sim}{I}_c \, \dot{\underset{\sim}{\omega}}\big/_{rel} + \underset{\sim}{\Omega} \times (\underset{\sim}{I}_c \underset{\sim}{\omega}) = \underset{\sim}{M}_c$$

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$\omega_d, \omega_0 = \text{const} \Rightarrow \dot{\underset{\sim}{\omega}}\big/_{rel} = \underset{\sim}{0}$

$I_x = \frac{1}{2} m R^2 \qquad I_y = I_z = \frac{1}{4} m R^2$

$$\underset{\approx}{I}_c \underset{\sim}{\omega} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \omega_d \\ 0 \\ \omega_0 \end{pmatrix} = \begin{pmatrix} I_x \omega_d \\ 0 \\ I_z \omega_0 \end{pmatrix}$$

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$$\begin{array}{rl} & \underset{\sim}{\Omega}\,(0, 0, \omega_0) \\ \times & \underset{\approx}{I}_c \underset{\sim}{\omega}\,(I_x \omega_d, 0, I_z \omega_0) \\ \hline & (0, I_x \omega_0 \omega_d, 0) \end{array} \Rightarrow$$

$$|\underset{\sim}{M}_c| = M_{cy} = I_x \omega_0 \omega_d$$

$$= 0.675\, \omega_0 \quad [kg \cdot m^2 \cdot s^{-2}]$$