CES 389C/EM 397 INTRODUCTION TO MATHEMATICAL MODELING IN SCIENCE AND ENGINEERING
Exam 2, Dec 7, 2012

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - expected value and standard deviation of a random variable,
   - discrete and continuous spectrum of a self-adjoint operator,
   - group velocity,
   - incompatible observables
   - Bosons and Fermions.

   See the class notes.

2. Discuss the statistical and *generalized* statistical interpretation of wave function (15 points).

   See the class notes.

3. State and prove the *Uncertainty Principle* (15 points).

   See the class notes.

4. Consider the 1D time-independent Schrödinger equation with the potential corresponding to the harmonic oscillator:
$$V(x) = \frac{1}{2}m\omega^2 x^2$$

   (a) Arrive at the eigenvalue problem
$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$
   where $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ and $K = \frac{2E}{\hbar\omega}$.

   (b) Use ansatz
$$\psi(\xi) = h(\xi)e^{-\frac{\xi^2}{2}}$$
   to arrive at the equation for function h,
$$h'' - 2\xi h' + (k-1)h = 0$$

   (c) Discuss the application of Frobenius method to arrive at the Hermite polynomials. I would like to see the recursion formula, forget about the scaling factors...

   (d) Produce formulas for the energies corresponding to the ground and excited states.

   (20 points).

   See the class notes.

5. (a) Prove that for a particle in a general potential $V = V(\boldsymbol{r})$,

$$\frac{d}{dt}\langle \boldsymbol{L} \rangle = \langle \boldsymbol{N} \rangle$$

where $\boldsymbol{N} = \boldsymbol{r} \times (-\boldsymbol{\nabla} V)$. This is the analog of Ehrenfest's theorem for the angular momentum principle.

You can refer to the general Ehrenfest theorem but you can also compute it directly.

Wave function satisfies the Schrödinger equation:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\psi_{,ll} + V\psi$$

If you conjugate it, you get the equation satisfied by the conjugate ψ^* (do not miss the change of sign on the left-hand side),

$$-i\hbar\frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi^*_{,ll} + V\psi$$

Use the two equations to compute the time derivative of the expected valus of the angular momentum,

$$\begin{aligned}
\frac{d}{dt}\int \psi^*(-i\hbar\epsilon_{ijk}x_j\psi_{,k}) &= \int(-i\hbar\frac{\partial \psi^*}{\partial t})\epsilon_{ijk}x_j\psi_{,k} + \int \psi^*\epsilon_{ijk}x_j(-i\hbar\frac{\partial \psi}{\partial t})_{,k} \\
&= \int(-\frac{\hbar^2}{2m}\psi^*_{,ll} + V\psi^*)\epsilon_{ijk}x_j\psi_k + \int \psi^*\epsilon_{ijk}x_j(\frac{\hbar^2}{2m}\psi_{,ll} - V\psi)_{,k} \\
&= \frac{\hbar^2}{2m}\int(-\psi^*_{,ll}\epsilon_{ijk}x_j\psi_{,k} + \psi^*\epsilon_{ijk}x_j\psi_{,ll}) + \int \psi^*\epsilon_{ijk}x_j(-V_{,k})\psi
\end{aligned}$$

Integrate the first term by parts to show that it vanishes.

(b) Show that

$$\frac{d}{dt}\langle \boldsymbol{L} \rangle = \boldsymbol{0}$$

for any *spherical symmetric potential*, $V = V(r)$. This covers the case of hydrogen atom (Coulomb's potential).

Angular momentum wrt to the origin for a central force is zero.

(20 points).

6. Discuss the main ideas of the perturbation theory for time-independent problems (10 points).

See the class notes.