CSE386C
METHODS OF APPLIED MATHEMATICS
Fall 2019, Final Exam, 9:00-noon, Fri, Dec 13, ACES 6.304

1. Let T be a compact operator from a Hilbert space U into a Hilbert space V.
   (a) Define the notion of compact operators.
   (b) Show that T^*T and TT^* are compact, self-adjoint, positive semi-definite operators from U (V, resp.) into itself.
   (c) Prove that all eigenvalues of a self-adjoint operator are real.
   (d) Prove that T^*T and TT^* have identical non-negative eigenvalues and derive a relation between the corresponding eigenspaces.

(25 points)

(a) See the book

(b) Composition of a compact and continuous operator (in any order) is compact. We have,

$$(T^*Tu, u)_U = \underbrace{(Tu, Tu)_V}_{\geq 0} = (u, T^*Tu)$$

and the same argument holds for TT^*.

(c) Let $\lambda \neq 0$ be an eigenvalue of a self-adjoint operator A from a Hilbert space U into itself, and u the corresponding eigenvector. We have,

$$\lambda(u, u) = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \bar{\lambda}(u, u) \, .$$

Hence

$$(\lambda - \bar{\lambda})(u, u) = 0 \quad \Rightarrow \quad \lambda - \bar{\lambda} = 0 \quad \Rightarrow \quad \lambda \in \mathbb{R} \, .$$

(d) Let (λ, u) be an eigenpair for T^*T, $\lambda \neq 0$,

$$T^*Tu = \lambda u \, .$$

Apply T to both sides of the equation to get,

$$TT^* \, Tu = \lambda Tu$$

which proves that (λ, Tu) is an eigenpair for TT^*. Converssely, if (λ, v) is an eigenpair for TT^* then (λ, T^*v) is an eigenpair for T^*T. Let

$$U_\lambda := \mathcal{N}(\lambda - T^*T), \qquad V_\lambda := \mathcal{N}(\lambda - TT^*)$$

be the eigenspaces corresponding to λ. The first property above proves that T sets U_λ into V_λ,

$$T(U_\lambda) \subset V_\lambda \, .$$

Let $v \in V_\lambda$, i.e.

$$TT^*v = \lambda v \quad \Rightarrow \quad v = T(\lambda^{-1}T^*v)\, ,$$

i.e. there exists an $u \in U_\lambda$, namely, $u = \lambda^{-1}T^*v$ such that $v = Tu$. In other words,

$$T(U_\lambda) = V_\lambda \, .$$

By the same argument,

$$T^*(V_\lambda) = U_\lambda \, .$$

Finally,

$$T^*Tu = 0 \quad \Rightarrow \quad T^*v = 0 \text{ for } v = Tu \quad \Rightarrow \quad TT^*v = 0\, .$$

i.e. $T(\mathcal{N}(T^*T)) \subset \mathcal{N}(TT^*)$. Similarly, $T^*(\mathcal{N}(TT^*)) \subset \mathcal{N}(T^*T)$.
Note that $\mathcal{N}(T^*T) = \mathcal{N}(T)$ and $\mathcal{N}(TT^*) = \mathcal{N}(T^*)$.

2. (a) Define discrete, residual and continuous spectrum for an operator $A : U \supset D(A) \to U$ where U is a Hilbert space.

   (b) Determine spectrum of operator A where

$$U = L^2(\mathbb{R}) \quad D(A) = H^1(\mathbb{R}) \quad Au = \frac{du}{dx} + u$$

   Hint: Use Fourier transform.

(25 points)

This is a slight modification of the example discussed in the book for $Au = u'$. Direct computations using Fourier transform reveal that there is neither point nor residual spectrum.

The continuous spectrum consists of the line $\lambda = 1 + i\xi, \xi \in \mathbb{R}$.

3. Consider a first order operator A in $L^2(0,1)$,

$$D(A) = \{u \in H^1(0,1) \,:\, u(0) = u(1) = 0\} \qquad Au = u' - 2u$$

where the derivative is understood in the sense of distributions.

(a) Define a closed operator and prove that operator A is closed. You may use the fact that pointwise value $u(x), x \in [0,1]$ of $u \in H^1(0,1)$ represents a continuous functional.

(b) Determine the adjoint operator A^* and its null space.

(c) Prove that operator A is bounded below in $L^2(0,1)$.

(d) Discuss the well posedness of the problem:

$$u \in D(A), \qquad Au = f$$

with an appropriate right-hand side f.

(25 points)

See book for definitions.

Answers:

(a) Let $u_n \in D(A)$ and $(u_n, Au_n) \to (u, v)$, i.e. $u_n \to u$, $Au_n \to v$, all convergence understood in the L^2-sense. Consequently, $u'_n \to v + 2u$. By definition,

$$\int u'_n \phi = -\int u_n \phi' \quad \forall \phi \in C_0^\infty(0,1)$$

Passing to the limit on both sides, we get

$$\int (v + 2u)\phi = -\int u\phi' \quad \forall \phi \in C_0^\infty(0,1)$$

which proves that $v + 2u = u'$ in the sense of distributions.
Thus $u' = v + 2u \in L^2(0,1)$ and, therefore, $u_n \to u$ also in $H^1(0,1)$ which in turn implies that $u(0) = u(1) = 0$. Consequently, $u \in D(A)$, and $v = u' - 2u = Au$ as required.

(b) Integration by part argument gives:

$$\begin{aligned} D(A^*) &= H^1(0,1) \quad A^*v = -v' - 2v \\ \mathcal{N}(A^*) &= \mathbb{R}e^{-2x} := \{ce^{-2x} \,:\, c \in \mathbb{R}\} \end{aligned}$$

(c) We have,

$$\|Au\|^2 = \int_0^1 (u'-2u)^2 = \int_0^1 (u')^2 - 4\int_0^1 uu' + 4\int_0^1 u^2 = \int_0^1 (u')^2 4\int_0^1 u^2 \geq \underbrace{(C_P+4)}_{=:\alpha^2} \|u\|^2$$

where C_P is the Poincarè constant. Note that

$$\int_0^1 2uu' = \int_0^1 (u^2)' = u^2|_0^1 = 0 \quad \text{for } u \in D(A)\,.$$

(d) For every right hand side $f \in L^2(0,1)$ that satisfies the compatibility condition:

$$\int_0^1 f(x)e^{-2x}\,dx = 0\,,$$

the problem has a unique solution that depends continuously upon the data

$$\|u\| \leq \alpha^{-1}\|f\|$$

where α is the boundedness below constant.

4. Consider the "ultraweak" variational formulation of the previous problem,

$$\begin{cases} u \in U := L^2(0,1) \\ \underbrace{\int_0^1 u\, A^* v\, dx}_{b(u,v)} = \underbrace{\int_0^1 f v\, dx}_{l(v)} \quad \forall v \in V := H^1(0,1) \end{cases}$$

where A^* denotes the formal adjoint of A, $A^* v = -v' - 2v$.

(a) Define operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u,v)$.

(b) Use Babuška-Nečas Theorem and results from the previous problem to investigate the well-posedeness of the problem.

Hint: Can you relate the inf-sup constant for this problem with the boundedness below (Friedrichs) constant of operator A from the previous problem ? (25 points)

There are two operators associated with the bilinear form:

$$B : L^2(0,1) \to (H^1(0,1))'$$

$$B' : H^1(0,1) \to L^2(0,1) \sim (L^2(0,1))'$$

Due to reflexivity of Hilbert space, operator B' can be identified with the transpose of B. The whole point of this exercise is to realize that transpose B' *coincides* with the adjoint A^* discussed in the previous problem. Direct application of Cauchy-Schwartz inequality shows that both forms: $b(u,v)$ and $l(v)$ are continuous. Finally, the Closed Range Theorem for continuous operators implies that

$$\gamma = \inf_{u \in L^2} \sup_{v \in H^1} \frac{|\int_0^1 u(-v' - 2v)|}{\|u\|_{L^2} \, \|v\|_{H^1}} = \inf_{[v] \in H^1/\mathcal{N}(B')} \sup_{u \in L^2} \frac{|\int_0^1 u(-v' - 2v)|}{\|u\|_{L^2} \, \|[v]\|_{H^1/\mathcal{N}(B')}} \, .$$

Since $B' = A^*$, the right-hand side coincides with the boundedness below (Friedrichs) constant for the quotient operator corresponding to A^*. But, by the Closed Range Theorem for closed operators, this constant is equal exactly to constant α discussed in the previous problem. Consequently, there is no need to prove anything new. Application of Babuška-Nečas Theorem implies that, for any right-hand side f satisfying the compatibility condition, we have a unique solution that depends continuously upon the data. The subtle difference between the strong and (ultraweak) variational formulations is the regularity. The ultraweak formulation may accommodate "distributional loads":

$$l \in \left(H^1(0,1)\right)'$$

with the continuous dependence upon data modified accordingly:

$$\|u\| \le \gamma^{-1} \|l\|_{(H^1(0,1))'} \, .$$