CSE386C
METHODS OF APPLIED MATHEMATICS
Fall 2014, Exam 3

1. Define the following notions and provide a non-trivial example (2+2 points each):

   - (topological) transpose of a continuous operator,
   - (topological) transpose of a closed operator,
   - orthogonal complement of a subspace in a Hilber space,
   - orthonormal basis in a Hilbert space,
   - Riesz operator.

   See the book.

2. State and prove *three* out of the four theorems (10 points each):

   - Properties of the transpose of a continuous operator (Prop. 5.16.1)
   - Characterization of injective operators with closed range (Thm. 5.17.1)
   - Completness of qoutient Banach space (Lemma 5.17.1)
   - The Orthogonal Decomposition Theorem (Thm. 6.2.1)

   See the book.

3. Let X be a Banach space, and $P : X \to X$ be a continuous linear projection, i.e., $P^2 = P$. Prove that the range of P is closed. (10 points)

   Let $u_n \in \mathcal{R}(P), u_n \to u$. We need to show that $u \in \mathcal{R}(P)$ as well. Let $v_n \in X$ be such that $u_n = Pv_n$. Then $Pu_n = P^2 v_n = Pv_n = u_n \to Pu$. By the uniqueness of the limit, it must be $u = Pu$. Consequently, u is the image of itself and must be in the range of the projection.

4. Let $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ be an orthonormal family in a Hilbert space V. Prove that the following conditions are equivalent to each other.

   (i) $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ is an orthonormal basis, i.e., it is maximal.

   (ii) $\boldsymbol{u} = \sum_{n=1}^{\infty} (\boldsymbol{u}, \boldsymbol{e}_n)\, \boldsymbol{e}_n \qquad \forall\, \boldsymbol{u} \in V.$

   (iii) $(\boldsymbol{u}, \boldsymbol{v}) = \sum_{n=1}^{\infty} (\boldsymbol{u}, \boldsymbol{e}_n)\, \overline{(\boldsymbol{v}, \boldsymbol{e}_n)}.$

(iv) $\|\boldsymbol{u}\|^2 = \sum_{n=1}^{\infty} |(\boldsymbol{u}, \boldsymbol{e}_n)|^2$.

(15 points)

(i)⇒(ii). Let

$$\boldsymbol{u}_N := \sum_{j=1}^{N} u_j \boldsymbol{e}_j, \quad \boldsymbol{u}_N \to \boldsymbol{u}$$

Multiply both sides of the equality above by $\boldsymbol{e}_i$, and use orthonormality of $\boldsymbol{e}_j$ to learn that

$$u_i = (\boldsymbol{u}_N, \boldsymbol{e}_i) \to (\boldsymbol{u}, \boldsymbol{e}_i) \text{ as } N \to \infty$$

(ii)⇒(iii). Use orthogonality of $\boldsymbol{e}_i$ to learn that

$$(\boldsymbol{u}_N, \boldsymbol{v}_N) = \sum_{i=1}^{N} u_i \overline{v}_i = \sum_{i=1}^{N} (\boldsymbol{u}, \boldsymbol{e}_i)\, \overline{(\boldsymbol{v}, \boldsymbol{e}_i)} \to \sum_{i=1}^{\infty} (\boldsymbol{u}, \boldsymbol{e}_i)\, \overline{(\boldsymbol{v}, \boldsymbol{e}_i)}$$

(iii)⇒(iv). Substitute $\boldsymbol{v} = \boldsymbol{u}$.

(iv)⇒(i). Suppose, to the contrary, the $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots\}$ can be extended with a vector $\boldsymbol{u} \neq \boldsymbol{0}$ to a bigger orthonormal family. Then $\boldsymbol{u}$ is orthogonal with each $\boldsymbol{e}_i$ and, by property (iv), $\|\boldsymbol{u}\| = 0$. So $\boldsymbol{u} = \boldsymbol{0}$, a contradiction.

5. Consider an elementary boundary-value problem:

$$\begin{cases} u(0) = 0 \\ u' + u = f \end{cases}$$

Use elementary means (variation of a constant) to derive the explicit formula for the solution,

$$u(x) = \int_0^x e^{(s-x)} f(s)\, ds\,.$$

Use the formula then to demonstrate that the operator

$$A \,:\, L^2(0,1) \supset D(A) \to L^2(0,1)$$

where

$$D(A) = \{u \in H^1(0,1) \,:\, u(0) = 0\}, \quad Au := u' + u$$

is bounded below. (10 points).

The first part is elementary. The second follows from Cauchy-Schwarz inequality:

$$\begin{aligned}\int_0^1 |u(x)|\,dx &= \int_0^1 |\int_0^x e^{(s-x)} f(s)\,ds|^2\,dx \\ &\le \int_0^1 \int_0^x \underbrace{|e^{(s-x)}|^2}_{\le e^2} \int_0^1 |f(s)|^2\,ds\,dx \\ &\le \frac{e^2}{2}\int_0^1 |f(s)|^2\,ds\end{aligned}$$

6. Analyze well posedness of the variational problem;

$$\begin{cases} u \in H^1(0,1),\ u(0)=0 \\ \displaystyle\int_0^1 (u'+u)v\,dx = \int_0^1 fv\,dx \quad \forall v \in L^2(0,1)\end{cases}$$

Hint: Use the result from the previous problem. (15 points)

Check the assumptions of Babuška- Nečas Theorem. The only non-trivial condition is the inf-sup condition. But this follows from the previous problem. Indeed,

$$\sup_v \frac{|\int_0^1 (u'+u)v|}{\|v\|_{L^2}} = \|u'+u\|_{L^2} \ge c\|u\|_{L^2}$$

At the same time,

$$\|u'\|_{L^2} \le \|u'+u\|_{L^2} + \|u\|_{L^2} \le (1+c^{-1})\|u'+u\|_{L^2}$$

Combining the two inequalities, we prove the inf-sup condition.