CSE386C
METHODS OF APPLIED MATHEMATICS
Fall 2019, Exam 1

1. Define the following notions and provide a non-trivial example (2+2 points each):

   - (General) topological vector space,
   - Locally convex topological vector space,
   - Minkowski functional,
   - Norm of a continuous, linear map,
   - Open map.

   See the book.

2. State and prove *three* out of the four theorems (10 points each):

   - Properties of Minkowski Set (Prop. 5.2.2),
   - Characterization of convergence in Schwartz test space (Prop. 5.3.2),
   - Uniform Boundedness Theorem (Thm. 5.8.1),
   - Characterization of Continuous Linear Operators in Normed Spaces (Prop. 5.6.1).

   See the book.

3. Show that each of the seminorms inducing a locally convex topology is *continuous* with respect to this topology (10 points).

   This is a direct consequence of the way we define neighborhoods in the l.c. topology. Let p_ι be one of the seminorms defining the topology and $\epsilon > 0$. The set:

   $$M(\{\iota\}, \epsilon) := \{f \,:\, |p_\iota(f)| < \epsilon\}$$

   is, by definition, a neighborhood of 0 and, trivially,

   $$f \in M(\{\iota\}, \epsilon) \quad \Rightarrow \quad |p_\iota(f)| < \epsilon\,.$$

4. Let X and Y be two arbitrary topological vector spaces. Show that a linear transformation $A \in L(X, Y)$ is continuous iff it is continuous at 0. (10 points).

   This is a direct consequence of the way we set up neighborhoods in a t.v.s. If $\mathcal{B}_0$ is the base of neighborhoods of 0 then $x + \mathcal{B}_0 =: \mathcal{B}_x$ is the base of neighborhoods of x. Let $y + B \in y + \mathcal{B}_0$

be an arbitrary neighborhood of $y = Ax \in Y$. If A is continuous at 0 then there exists a neighborhood $C \in \mathcal{B}_0$ such that

$$A(C) \subset B \,.$$

It follows that

$$A(x + C) = Ax + A(C) \subset y + B \,.$$

5. Consider space $C([0,1])$ with the *pointwise convergence topology* introduced by seminorms

$$p_x(f) := |f(x)|, \quad x \in [0,1] \,.$$

Show that the seminorms satisfy the *axiom of separation* which implies that the l.c.t.v.s. is well-defined. Demonstrate that $f_n \to f$ in that topology if and only if

$$f_n(x) \to f(x) \quad \forall x \in [0,1] \,.$$

(10 points).

The axiom of separation is trivially satisfied. If $f \in C([0,1])$, $f \neq 0$ then there exists $x \in [0,1]$ such that $f(x) \neq 0$. Consequently,

$$p_x(f) = |f(x)| > 0 \,.$$

By linearity, it is sufficient to show the result for $f = 0$, i.e.

$$f_n \to 0 \text{ in the topology} \quad \Leftrightarrow \quad f_n(x) \to 0, \, \forall x \in [0,1] \,.$$

If $f_n \to 0$ then $f_n(x) = p_x(f_n) \to 0$ as well since p_x is continuous in the topology (see the problem above). Conversely, let

$$B(I_0, \epsilon) = \{f \in C([0,1]) \, : \, |f(x)| < \epsilon, \, x \in I_0\}, \quad I_0 \subset [0,1], I_0 \text{ finite}$$

be an arbitrary neighborhood of 0. For each $x \in I_0$ there exists N_x such that

$$n \geq N_x \quad \Rightarrow \quad |f_n(x)| < \epsilon \,.$$

Set

$$N := \max_{x \in I_0} N_x \,.$$

Note that $\# I_0 < \infty$ implies that N is well defined. Then,

$$n \geq N \quad \Rightarrow \quad |f_n(x)| < \epsilon \quad \forall x \in I_0 \quad \Rightarrow \quad f_n \in B(I_0, \epsilon) \,.$$

6. Let A be a matrix representing a linear map from $(\mathbb{R}^n, \|\cdot\|_\infty)$ into $(\mathbb{R}^n, \|\cdot\|_1)$. Show that

$$\|A\|_{\infty,1} \leq \sum_{ij} |A_{ij}|$$

but construct a counterexample for the equality. (20 points)

Let $|x_j| \leq 1,\ j = 1, \ldots, n$. Then $|\sum_{j=1}^N A_{ij}x_j| \leq \sum_{j=1}^n |A_{ij}|$. Consequently,

$$\|\boldsymbol{Ax}\|_1 = \sum_{i=1}^n |\sum_{j=1}^n A_{ij}x_j| \leq \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|$$

Take now $n = 2$, and consider matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So,

$$\boldsymbol{Ax} = (x_1 + x_2, x_1 - x_2)^T$$

and

$$\|\boldsymbol{Ax}\|_1 = |x_1 + x_2| + |x_1 - x_2|$$

The maximum,

$$\max_{\|\boldsymbol{x}\|_\infty \leq 1} \|\boldsymbol{Ax}\|_1$$

is attained on the boundary $\|\boldsymbol{x}\|_\infty = 1$. Also,

$$\|\boldsymbol{A}(-\boldsymbol{x})\|_1 = \| - \boldsymbol{Ax}\|_1 = \|\boldsymbol{Ax}\|_1$$

so it is sufficient to consider only the following two cases.

Case: $|x_1| \leq 1,\ x_2 = 1$.

$$\|\boldsymbol{Ax}\|_1 = |x_1 + 1| + |x_1 - 1| = x_1 + 1 + 1 - x_1 = 2$$

Case: $x_1 = 1,\ |x_2| \leq 1$.

$$\|\boldsymbol{Ax}\|_1 = |1 + x_2| + |1 - x_2| = 2$$

Consequently,

$$\|\boldsymbol{A}\|_{\infty,1} = 2 < \sum_{i,j} |A_{ij}| = 4$$