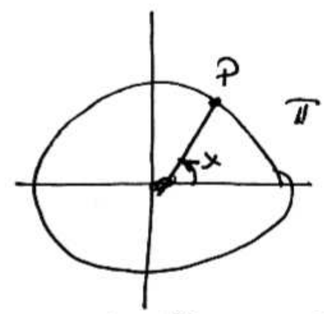
FOURIER SERIES

The spaces $C(\mathbb{T})$ & $L^2(\mathbb{T})$

Let f be a cont function on $\mathbb{R}$.
We say that f is 2π-periodic if $f(x)=f(x+2\pi)$ $\forall x\in\mathbb{R}$.
A 2π-periodic function is uniquely determined by its values on $I=[-\pi,\pi)$ (or any other interval of length 2π, $(0,2\pi]$, $[5,5+2\pi)$).
It is convenient to think of f as a function defined on the unit circle $\mathbb{T}$.
We typically parameterize $\mathbb{T}$ using the interval $[-\pi,\pi]$ (with the endpoints identified).



$C(\mathbb{T})=\{$The set of cont functions $f:[-\pi,\pi]\to\mathbb{C}$ s.t. $f(-\pi)=f(\pi)\}$.
Next we define the $L^2(\mathbb{T})$ norm; for $f\in C(\mathbb{T})$, set

$$\|f\|_{L^2}=\left[\int_{\mathbb{T}}|f(x)|^2dx\right]^{1/2}$$

and define the ~~Banach Space~~ Hilbert Space $L^2(\mathbb{T})$ as the closure of $C(\mathbb{T})$ under the L^2-norm.
It can be proven that
$L^2(\mathbb{T})=$ The set of Lebesgue measurable functions $f:\mathbb{T}\to\mathbb{C}$ s.t. $\int_{\mathbb{T}}|f(x)|^2dx<\infty$.
Actually, this is not quite precise; ~~we have to identify all fn~~ if f & g are functions s.t. $\int|f(x)-g(x)|^2=0$, then f and g are considered to be identical in $L^2(\mathbb{T})$. Thus $L^2(\mathbb{T})$ consists of a collection of equivalence classes.

A basis for $L^2(\mathbb{T})$

For $n \in \mathbb{Z}$, define $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} = \frac{1}{\sqrt{2\pi}}(\cos(nx) + i\sin(nx))$.

We have $\langle e_n, e_n \rangle = \cdots = 1$

$\langle e_n, e_m \rangle = \cdots = 0$ if $n \neq m$,

so $(e_n)_{n=-\infty}^{\infty}$ is an ON-set.

We will next prove that (e_n) is in fact a basis for $L^2(\mathbb{T})$.

We will use the following fact:

<u>Prop^n</u> Let H be a H.S., and let $(\varphi_n)_{n=1}^{\infty}$ be an ON-set. Then (φ_n) is a basis $\Leftrightarrow$ The set of finite linear comb^{ns} of (φ_n)'s is dense.

<u>Proof:</u> Homework

Set $\mathcal{P}_N = \operatorname{span}(e_n)_{n=-N}^{N} = \{f = \sum_{n=-N}^{N} \alpha_n e^{inx}\} = \{f = \sum_{n=0}^{N} (\alpha_n \cos(nx) + \beta_n \sin(nx))\}$

$\mathcal{P} = \bigcup_{N=1}^{\infty} \mathcal{P}_N$

$\mathcal{P}_N$ is called the set of all trig. polynomials of degree $\leq N$.

$\mathcal{P}$ — '' ————————.

<u>Thm</u> $\mathcal{P}$ is dense in $C(\mathbb{T})$

<u>Corollary</u>: (e_n) is a basis for $L^2(\mathbb{T})$

Proof of Corollary: According to the prop^n above, it is sufficient to prove that $\mathcal{P}$ is dense in $L^2(\mathbb{T})$.

First note that if $\varphi \in C(\mathbb{T})$, then

$$\|\varphi\|_2 = \left(\int_{-\pi}^{\pi} |\varphi|^2\right)^{1/2} \leq \left(2\pi \|\varphi\|_u^2\right)^{1/2} = \sqrt{2\pi}\, \|\varphi\|_u. \qquad (1)$$

Fix $f \in L^2(\mathbb{T})$ & $\varepsilon > 0$.

By def^n, $\exists\ \varphi \in C(\mathbb{T})$ s.t. $\|\varphi - f\|_2 < \varepsilon/2$. $\qquad (2)$

According to the thm, $\exists\ p \in \mathcal{P}$ s.t. $\|p - \varphi\|_u \leq \dfrac{\varepsilon}{2\sqrt{2\pi}}$. $\qquad (3)$

Combining (1), (2), & (3), we obtain

$$\|f - p\|_2 \leq \|f - \varphi\|_2 + \|\varphi - p\|_2 \leq \underbrace{\|f - \varphi\|_2}_{<\varepsilon/2} + \sqrt{2\pi}\, \underbrace{\|\varphi - p\|_u}_{<\varepsilon/2\sqrt{2\pi}} < \varepsilon.$$

In order to prove the Theorem, we need two technical tools: Convolutions & Approximate Identities.

Def^n Given $f, g \in C(\mathbb{T})$, define the function $f * g$, the convolution between f & g by

$$[f * g](x) = \int_{\mathbb{T}} f(x-y)\, g(y)\, dy$$

Lemma
- $f * g = g * f \quad \left(\Leftrightarrow \int f(x-y) g(y)\, dy = \int f(y) g(x-y)\, dy\right)$
- $f * g \in C(\mathbb{T})$

Def^n A sequence $(\varphi_n)_{n=1}^{\infty} \in C(\mathbb{T})$ is called an <u>approximate identity</u> if

(i) $\varphi_n(x) \geq 0 \quad \forall x, n$

(ii) $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1 \quad \forall n$

(iii) $\forall \delta > 0 \quad \int_{|x| \geq \delta} |\varphi_n(x)|\,dx \to 0$ as $n \to \infty$.

In other words, φ_n is a sequence of functions whose mass concentrate at the origin.

Lemma: Suppose that $(\varphi_n)_{n=1}^{\infty}$ is an approx identity & that $f \in C(\mathbb{T})$, then $\varphi_n * f \to f$ uniformly.

Proof: Set $f_n = \varphi_n * f$.

(ii) $\Rightarrow \quad f(x) = \int \varphi_n(y) f(x)\,dy \qquad (1)$

By def^n: $\quad f_n(x) = \int \varphi_n(y) f(x-y)\,dy \qquad (2)$

(1) & (2) $\Rightarrow \quad f(x) - f_n(x) = \int \varphi_n(y)[f(x) - f(x-y)]\,dy$.

Fix $\varepsilon > 0$.

Since f is uniformly cont, $\exists\, \delta$ s.t. $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$.

~~Set M=~~ Now,

$$|f(x) - f_n(x)| \leq \int_{|x| \leq \delta} \varphi_n(y)\,|f(x) - f(x-y)|\,dy + \int_{|x| > \delta} \varphi_n(y) \underbrace{|f(x) - f(x-y)|}_{\leq 2\|f\|}\,dy$$

$$\leq \varepsilon \underbrace{\int_{|x| \leq \delta} \varphi_n(y)\,dy}_{\leq 1} + 2\|f\| \int_{|x| > \delta} \varphi_n(y)\,dy$$

Take sup over x: $\quad \|f - f_n\| \leq \varepsilon + 2\|f\| \int_{|x| > \delta} \varphi_n(y)\,dy$

Take limsup as $n \to \infty$ $\quad \limsup_{n \to \infty} \|f - f_n\| \leq \varepsilon$

Since ε is arbitrary: $\lim_{n \to \infty} \|f - f_n\| = 0$

Proof of Thm Fix $f \in C(\mathbb{T})$.

We will construct $f_n \in \mathcal{P}$ s.t. $f_n \to f$ uniformly.

Set $\varphi_n(x) = c_n (1+\cos x)^n$ where $c_n = \dfrac{1}{\int_{\mathbb{T}} (1+\cos x)^n dx}$

Then $(\varphi_n)_{n=1}^{\infty}$ is an approximate identity

and so if we set $f_n = \varphi_n * f$, we have $f_n \to f$ uniformly.

It remains to prove that $f_n \in \mathcal{P}$. We have

$$\varphi_n = c_n(1+\cos x)^n = c_n\left(1 + \tfrac{1}{2}e^{ix} + \tfrac{1}{2}e^{-ix}\right)^n = \sum_{j=-n}^{n} \beta_j^{(n)} e^{ijx},$$

for some $\beta_j^{(n)} \in \mathbb{C}$. Thus

$$f_n(x) = \int \varphi_n(x-y) f(y)\,dy = \sum_{j=-n}^{n} \beta_j^{(n)} \int e^{ij(x-y)} f(y)\,dy =$$

$$= \sum_{j=-n}^{n} \beta_j^{(n)} e^{ijx} \int e^{-ijy} f(y)\,dy \in \mathcal{P}_n.$$

To sum up:

Given any $f \in L^2(\mathbb{T})$, we have $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{inx}}{\sqrt{2\pi}}$

where $\alpha_n = \langle e_n, f\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$.

unitary

The map $\mathcal{F} : L^2(\mathbb{T}) \to l^2(\mathbb{Z}) : f \mapsto (\alpha_n)_{n=-\infty}^{\infty}$

~~is an isometric. It is unitary (a H.S. isomorphism).~~

is called a Fourier transform.

~~The seq $(\alpha_n)_{n=-\infty}^{\infty}$ is the Fourier series of f.~~