APPM5450 — Applied Analysis: Section exam 2
1:00pm – 1:50pm, March 17, 2017.

Hand in solutions to Problems 1 – 4 no later than 1:50pm. Closed books for this part.

*Name:*____

Problem 1: (10p) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$.

(a) (5p) Define the *spectrum* $\sigma(A)$.

(b) (5p) Suppose that A is skew-adjoint and that $\|A\| = 2$. Are there any complex numbers λ for which you can say for sure that $A - \lambda I$ is one-to-one and onto?

Problem 2: (10p) Let $T \in \mathcal{S}^*(\mathbb{R})$ be defined via $T(\varphi) = \int_{-\infty}^{\infty} \log|x|\,\varphi(x)\,dx$. Specify the derivative of T. No motivation required.

Problem 3: (10p) No motivations required for these two problems.

(a) (5p) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$ be an operator that satisfies $A^2 = A = A^*$. The operator A is neither the zero or the identity operator. Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$.

(b) (5p) Let $H = L^2([0,\infty))$, and let $A \in \mathcal{B}(H)$ be defined by $[Au](x) = \arctan(x)\,u(x)$. Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$.

Problem 4: (10p) Consider the four sequences in $\mathcal{S}^*(\mathbb{R})$ given below. Specify which sequences are convergent. If the sequence is convergent, then specify the limit. No motivations required.

(a) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \sin(nx)$.

(b) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \begin{cases} n & \text{when } -1/n \le x \le 1/n, \\ 0 & \text{when } |x| > 1/n. \end{cases}$

(c) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \begin{cases} n^2 & \text{when } -1/n \le x \le 1/n, \\ 0 & \text{when } |x| > 1/n. \end{cases}$

(d) $(T_n)_{n=1}^{\infty}$ where $T_n(x) = \displaystyle\sum_{m=0}^{n} \frac{x^m}{m!}$.

(Two points per problem for correctly identifying whether the sequence is convergent or not. An additional two points total if the limits are given correctly.)

Solutions to problems 5 and 6 are due on Wednesday March 22 at 1pm. Open books.

Problem 5: (20p) Let H denote the Hilbert space $H = \ell^2(\mathbb{Z})$. In other words, a doubly indexed vector $x = \{x(n)\}_{n=-\infty}^{\infty}$ belongs to H iff $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Define $A \in \mathcal{B}(H)$ via:

$$[Ax](n) = x(n+1) - x(n-1), \qquad n \in \mathbb{Z}.$$

Let $F : L^2(\mathbb{T}) \to H$ denote the standard Fourier transform, and let F^{-1} denote its inverse. Define

$$B = F^{-1} A F$$

as an operator on $L^2(\mathbb{T})$.

(a) (5p) Determine the action of B on a function $u = u(t)$ in $L^2(\mathbb{T})$.

(b) (15p) Determine $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$, and $\sigma_{\mathrm{r}}(A)$.

Please motivate your answers.

Problem 6: (4×5p) For each of the four operators defined below, determine whether it is well-defined, and whether it is continuous.

(a) $A : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $A(\varphi) = \int_{\mathbb{R}} x^2 \, \varphi(x) \, dx$.

(b) $B : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $B(\varphi) = \int_{\mathbb{R}} x \left(\varphi(x)\right)^2 dx$.

(c) $C : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined via $[C(\varphi)](x) = x \, \varphi(x)$.

(d) $D : \mathcal{S}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})$ defined via $DT = \partial T$. (Just differentiation in the plain distributional sense.)

Please briefly motivate your answer in each case.

Hints: By "well-defined," we mean that the expression can be evaluated and that the result belongs to the stated set. For instance, the operator A is well-defined if the integral evaluates to a finite complex number for every $\varphi \in \mathcal{S}(\mathbb{R})$. Also, observe that you do not need to state whether the operator given is linear or not.