

Homework set 8 — APPM5450, Spring 2017

From the textbook: 11.1(b,c), 11.2, 11.6, 11.7 (important!), 11.8, 11.10.

Problem 1: Define a function f on \mathbb{R} via $f(x) = |x|$. Compute carefully the first and the second derivatives of f where the differentiation is understood in a distributional sense. (We did this in class, but please try to redo it without looking at your notes.)

Problem 2: Prove that if $f \in C^\infty(\mathbb{R}^d)$, and for every $\alpha \in \mathbb{Z}^d$, there exist finite C and N such that $|\partial^\alpha f(x)| \leq C(1 + |x|^N)$, then $f\varphi \in \mathcal{S}$ whenever $\varphi \in \mathcal{S}$. Moreover, prove that if $\varphi_n \rightarrow \varphi$ in \mathcal{S} , then $f\varphi_n \rightarrow f\varphi$ in \mathcal{S} .

Problem 3: Demonstrate that a tempered function is not necessarily of at most polynomial growth by constructing a continuous function f on \mathbb{R} such that

$$(1) \quad \int_{-\infty}^{\infty} |f(x)| dx < \infty,$$

but

$$(2) \quad \sup_{x \in \mathbb{R}} \frac{|f(x)|}{(1 + |x|^2)^{k/2}} = \infty, \quad \forall k \in \{0, 1, 2, \dots\}.$$

If you would like to make the problem slightly harder, then construct a function f that satisfies (2) and also

$$(3) \quad \int_{-\infty}^{\infty} (1 + |x|^2)^{k/2} |f(x)| dx < \infty, \quad \forall k \in \{0, 1, 2, \dots\}.$$

Problem 4 (optional): Let \mathcal{D} denote the linear space $C_c^\infty(\mathbb{R}^d)$. We define a topology on \mathcal{D} by saying that $\varphi_n \rightarrow \varphi$ if and only if there exists a compact set $K \subseteq \mathbb{R}^d$ such that $\text{supp}(\varphi_n) \subseteq K$ for all n , and $\|\partial^\alpha \varphi_n - \partial^\alpha \varphi\|_u \rightarrow 0$ for all $\alpha \in \mathbb{Z}_+^d$.

- Prove that \mathcal{D} is a linear subspace of \mathcal{S} .
- Prove that the set \mathcal{D} is not closed in the topology of \mathcal{S} .
- Prove that if $\varphi_n \rightarrow \varphi$ in \mathcal{D} , then $\varphi_n \rightarrow \varphi$ in \mathcal{S} .