

Homework set 7 — APPM5450, Spring 2017

From the text-book: 9.19, 9.20, 9.22. Optional: 9.21.

Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e^{(j)})_{j=1}^n$ be the canonical basis, and let A have the representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

in the canonical basis. We define the *Hilbert-Schmidt norm* of A as

$$\|A\|_{\text{HS}} = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

(a) Let $(\varphi^{(j)})_{j=1}^n$ be any ON-basis for H . Show that $\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|A\varphi^{(j)}\|^2$.

(b) Show that $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $\|G\|_{\text{HS}} = \|G\|$ and $\|H\|_{\text{HS}} = \sqrt{n}\|H\|$.

Problem 2: Let H be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that H has an ON-basis $(\varphi^{(j)})_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 < \infty.$$

Prove that if $(\psi^{(j)})_{j=1}^{\infty}$ is any other ON-basis, then

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \|A\psi^{(j)}\|^2.$$

Problem 3: Consider the linear space $L = \mathbb{R}^2$. Define for $x = (x_1, x_2) \in L$ the seminorms

$$p_1(x) = |x_1|, \quad p_2(x) = |x_2|.$$

Construct for $x \in L$, $j \in \{1, 2\}$, and $\varepsilon \in (0, \infty)$, the sets

$$\mathcal{B}_{x,j,\varepsilon} = \{y \in L : p_j(x - y) < \varepsilon\}.$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms $\{p_1\}$? Is it Hausdorff? What is the topology generated by the collection of semi-norms $\{p_1, p_2\}$? Is it Hausdorff?