Solutions for Homework 2 — APPM5450 — Spring 2017

Exercise 7.13: Set $I = [0, 1]$ and consider the equation

(1) $$i\, u_t = -u_{xx}, \qquad x \in I, \qquad t > 0,$$

for a complex valued function $u = u(x, t)$ with homogeneous boundary conditions,

(2) $$u(0, t) = u(1, t) = 0,$$

and initial condition

(3) $$u(x, 0) = f(x).$$

Set

$$e_n(x) = \sqrt{2}\, \sin(n\, x).$$

Then $(e_n)_{n=1}^{\infty}$ forms an ON-basis for $L^2(I)$. We look for a solution

(4) $$u(x, t) = \sum_{n=1}^{\infty} \alpha_n(t)\, e_n(x).$$

Inserting (4) into (1) and (3), we find that α_n must satisfy

$$i\, \alpha_n' = n^2\, \alpha_n, \qquad \alpha_n(0) = f_n,$$

where $f_n = (e_n, f)$. The solution is

$$\alpha_n(t) = f_n\, e^{-i\, n^2\, t}.$$

Since $|\alpha_n(t)| = |f_n|$ for any t, it follows directly from Parseval that

$$\|u(t)\|_{L^2(I)}^2 = \sum_{n=1}^{\infty} |\alpha_n(t)|^2 = \sum_{n=1}^{\infty} |f_n|^2 = \|f\|^2,$$

and that (using that the cosines also form an ON-set)

$$\|u_x(t)\|_{L^2(I)}^2 = \left\| \sum_{n=1}^{\infty} f_n\, e^{-i\, n^2\, t}\, n\, \sqrt{2} \cos(nx) \right\|_{L^2(I)}^2 = \sum_{n=1}^{\infty} |n\, f_n|^2 = \|f_x\|^2.$$

For a direct proof, set $v = \mathrm{Re}(u)$ and $w = \mathrm{Im}(u)$ so that $u = v + i\, w$. Then (1) takes the form

$$v_t = -w_{xx} \qquad w_t = v_{xx}.$$

Now

$$\frac{d}{dt} \int_0^1 |u|^2\, dx = \frac{d}{dt} \int_0^1 (v^2 + w^2)\, dx = 2 \int_0^1 (v_t\, v + w_t\, w)\, dx$$

$$= 2 \int_0^1 (-w_{xx}\, v + v_{xx}\, w)\, dx = 2 \int_0^1 (w_x\, v_x - v_x\, w_x)\, dx = 0.$$

The second to last step was partial integration where the boundary terms vanish due to (2). Analogously,

$$\frac{d}{dt} \int_0^1 |u_x|^2\, dx = \frac{d}{dt} \int_0^1 (v_x^2 + w_x^2)\, dx = 2 \int_0^1 (v_{xt}\, v_x + w_{xt}\, w_x)\, dx$$

$$= 2 \int_0^1 (-v_t\, v_{xx} - w_t\, w_{xx})\, dx = 2 \int_0^1 (-v_t\, w_t + w_t\, v_t)\, dx = 0.$$

In the second calculation we used differentiation, (2) takes the form

$$v_t(0, t) = v_t(1, t) = w_t(0, t) = w_t(1, t) = 0, \qquad t > 0.$$

Exercise 8.3: Let P and Q be orthogonal projections. Set $M = \operatorname{ran}(P)$ and $N = \operatorname{ran}(Q)$. TFAE:

(1) $M \subseteq N$
(2) $QP = P$
(3) $PQ = P$
(4) $||Px|| \leq ||Qx|| \qquad \forall x$
(5) $(x, Px) \leq (x, Qx) \qquad \forall x$

Proof:

(a) $\Rightarrow$ (b): Assume $M \subseteq N$. Then for any x, $Px \in M \subseteq N$, so $QPx = Px$.

(b) $\Rightarrow$ (a): Assume $QP = P$. Pick $y \in M$. Then $y = Px$ for some x. Then $Qy = QPx = Px = y$ so $y \in N$.

(a) $\Leftrightarrow$ (c):

$$
\begin{aligned}
M \subseteq N \qquad &\Leftrightarrow \qquad N^{\perp} \subseteq M^{\perp} \\
&\Leftrightarrow \qquad Py = 0 \quad \forall y \in N^{\perp} \\
&\Leftrightarrow \qquad P(I - Q)x = 0 \quad \forall x \\
&\Leftrightarrow \qquad P = PQ
\end{aligned}
$$

(c) $\Rightarrow$ (d): Assume $PQ = P$. Since $||P|| \leq 1$ we have $||Px|| = ||PQx|| \leq ||Qx||$ for any x.

(d) $\Rightarrow$ (a): Assume that (a) is false. Then there is an $x \in M \backslash N$. Since $x \in M$ we have $x = Px$ and so
$$||Px||^2 = ||x||^2 = ||Qx + (I - Q)x||^2 = ||Qx||^2 + ||(I - Q)x||^2.$$
Now observe that $||(I - Q)x|| > 0$ since $x \notin N$. Consequently,
$$||Qx||^2 = ||Px||^2 - ||(I - Q)x||^2 < ||Px||^2$$
so (d) cannot hold true.

(d) $\Leftrightarrow$ (e): Simply observe that $(x, Px) = (x, P^2x) = (Px, Px) = ||Px||^2$ and analogously $(x, Qx) = ||Qx||^2$.

Note: You may want to draw a diagram over the implications to convince yourself that all equivalencies have been proven.

Exercise 8.4: First we prove that $P_n \to I$ strongly. Fix any $x \in H$. Since $\bigcup_{n=1}^{\infty} \mathrm{ran}(P_n) = H$, we know that $x \in \mathrm{ran}(P_N)$ for some specific N. Then, since $\mathrm{ran}(P_n) \subseteq \mathrm{ran}(P_{n+1})$, we see that $x \in \mathrm{ran}(P_m)$ for any $m \geq N$. Consequently, $P_m x = x$ for any $m \geq N$ so $P_n x \to x$ (*very* rapidly!).

Next suppose that $||I - P_n|| \to 0$. Then there is some N such that $||I - P_N|| \leq 1/2$. Now observe that $I - P_N$ is itself an orthogonal projection (onto $\ker(P_N)$) so it can only have norms 0 and 1. It follows that $||I - P_N|| = 0$, which is to say that $P_N = I$. Since $H = \mathrm{ran}(P_N) \subseteq \mathrm{ran}(P_{N+1}) \subseteq \mathrm{ran}(P_{N+2}) \subseteq \cdots$ we see that $P_n = I$ for any $n \geq N$.

Problem 1: Let $T(t)$ denote the semigroup defined in Section 7.3 of the textbook. Prove that $T(t) \to I$ strongly as $t \searrow 0$. Prove that $T(t)$ does not converge in norm.

Solution: We consider a slightly more general problem. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for a Hilbert space H, and consider for $t \geq 0$ the operator

$$T(t)f = \sum_{n=1}^{\infty} f_n \, e^{-n^2 t} \, e_n.$$

We will show that as $t \searrow 0$, $T(t) \to I$ strongly but not in norm.

To show $T(t) \to I$ strongly, fix $f \in H$. Fix $\varepsilon > 0$. Set $f_n = (e_n, f)$ and pick N such that $\sum_{n=N+1}^{\infty} |f_n|^2 < \varepsilon^2$. Then by Parseval

$$||T(t)f - f||^2 = \sum_{n=1}^{N} \left| f_n \left(e^{-n^2 t} - 1 \right) \right|^2 + \sum_{n=N+1}^{\infty} \left| f_n \left(e^{-n^2 t} - 1 \right) \right|^2$$

$$\leq \sum_{n=1}^{N} \left| f_n \left(e^{-n^2 t} - 1 \right) \right|^2 + \sum_{n=N+1}^{\infty} 4 \, |f_n|^2 \leq \sum_{n=1}^{N} \left| f_n \left(e^{-n^2 t} - 1 \right) \right|^2 + 4\varepsilon^2.$$

Since only finitely many terms depend on t, we can now easily take the limit as $t \searrow 0$,

$$\limsup_{t \searrow 0} ||T(t)f - f||^2 \leq 4\,\varepsilon^2.$$

Since ε was arbitrary, we see that $\lim_{t \searrow 0} ||T(t)f - f|| = 0$.

To show that $T(t)$ does not converge to I in norm, we simply observe that for any $t > 0$

$$||T(t) - I|| \geq \sup_n ||\left(T(t) - I \right) e_n|| = \sup_n |e^{-n^2 t} - 1| = 1.$$

Problem 2: Suppose P is a projection on a Hilbert space H. TFAE:

(1) P is orthogonal, *i.e.* $\ker(P) = \operatorname{ran}(P)^{\perp}$.
(2) P is self-adjoint, *i.e.* $\langle P\,x,\,y\rangle = \langle x,\,P\,y\rangle \quad \forall x, y$.
(3) $\|P\| = 0$ or 1.

Proof:

(a) $\Rightarrow$ (b): Assume $\ker(P) = \operatorname{ran}(P)^{\perp}$. Pick any $x, y \in H$. Then

$$(Px,\,y) = (\underbrace{Px}_{\in\operatorname{ran}(P)},\,Py + \underbrace{(I-P)y}_{\in\ker(P)}) = (Px,\,Py) = (Px + (I-P)x,\,Py) = (x,\,Py).$$

(b) $\Rightarrow$ (c): Assume that (b) holds. Then for any x,

$$\|Px\|^2 = (Px,\,Px) = (P^2 x,\,x) = (Px,\,x) \le \|Px\|\,\|x\|,$$

so $\|P\| \le 1$. Obviously it is possible for $\|P\|$ to be zero. We need to prove that the only possible non-zero value of $\|P\|$ is one. To this end, note that if $P \ne 0$, then $\operatorname{ran}(P) \ne \{0\}$. Now observe that if x is a non-zero element in $\operatorname{ran}(P)$, we have $Px = x$ so $\|P\| \ge 1$.

(c) $\Rightarrow$ (a): Assume that (a) does not hold. Then there exist $x \in \operatorname{ran}(P)$ and $y \in \ker(P)$ such that $(x,\,y) \ne 0$. Set $\alpha = \overline{(x,\,y)}/|(x,\,y)|$ and $z = \alpha y$. Then $z \in \ker(P)$ and $(x,\,z) = |(x,\,y)| \in \mathbb{R}_+$. Set

$$w = x - z\,t.$$

Then $\|Pw\| = \|x\|$, and

$$\|w\|^2 = \|x\|^2 - 2t\,(x,\,z) + t^2\,\|z\|^2.$$

For small t, we see that $\|w\| < \|x\| = \|Pw\|$ so $\|P\| > 1$.

No solution is given for Problem 3 since the problem itself outlines precisely how to solve it — just fill in the details.