APPM5450 — Applied Analysis: Section exam 2 — Solutions
8:30 – 9:50, March 19, 2014. Closed books.

Problem 1: (12p) Let A be a self-adjoint bounded compact linear operator on a separable Hilbert space H. Which statements are necessarily true (no motivation required):

(a) H has an ON-basis of eigenvectors of A.

TRUE. (Note that when A has a null-space, you can just add an ON-basis for the null-space to the set of evecs associated with non-zero evals.)

(b) If $(e_n)_{n=1}^{\infty}$ is an ON-sequence, then $\lim_{n\to\infty} ||A\,e_n|| = 0$.

TRUE. You know that $e_n \rightharpoonup 0$, and since A is compact, it follows that $Ae_n \to 0$.

(c) For any $\lambda \in \mathbb{C}$, the subspace $\ker(A - \lambda I)$ is necessarily finite dimensional.

FALSE. If $\lambda = 0$, then the nullspace can be infinite dimensional.

(d) $\sigma_{\rm c}(A) = \emptyset$.

FALSE. The origin can be in the continuum spectrum.

(e) $\sigma_{\rm r}(A) = \emptyset$.

TRUE. Since A is self-adjoint.

(f) $||A||$ is necessarily an eigenvalue of A.

FALSE. It is possible that only $-||A||$ is an eval.

Problem 2: (12p) Let P be a projection on a Hilbert space H. Which of the following statements are necessarily correct (no motivation required):

(a) The spectral radius $r(P)$ is either precisely zero or precisely one.

TRUE. This follows from $r(P) = \lim_{n\to\infty} ||P^n||^{1/n}$ and $P^n = P$.

(b) $\sigma(P) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

TRUE. This follows from (a).

(c) $\sigma(P) \subseteq \mathbb{R}$.

(This problem is harder than I had intended. No points were deducted.)

(d) If P is orthogonal, then $\sigma(P) \subseteq \{0, 1\}$.

TRUE. You know that $\sigma(P)$ is real, and that $P = I$ on its range.

(e) If $||Px|| = ||x||$ for every $x \in H$, then P is necessarily the identity.

TRUE. Recall that $P = I$ on its range, and if $||Px|| = ||x||$ for every x, then ker$(P) = \{0\}$.

(f) If there exist $x \in \mathrm{ran}(P)$ and $y \in \ker(P)$ such that $\langle x,\, y\rangle \neq 0$, then $||P|| > 1$.

TRUE. See proof that P is S-A iff $||P|| = 1$ or 0.

Problem 3: (25p) Let H be a Hilbert space, and let A be a bounded linear operator on H, so that $A \in \mathcal{B}(H)$.

(a) Define the *resolvent set* $\rho(A)$.
(b) Prove that $\rho(A)$ is an open set.

See course notes for solution.

Problem 4: (25p) Define a map $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ via

$$T(\varphi) = \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} \frac{1}{x}\varphi(x)\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx \right).$$

Prove that T is a continuous functional on $\mathcal{S}$. (You do not need to prove linearity.) What can you say about the order of T?

Note: Recall that the *order* of a distribution is the lowest number m for which a bound of the form $|T(\varphi)| \leq C \sum_{\ell \leq k} \sum_{|\alpha| \leq m} ||\varphi||_{\ell,\alpha}$ holds.

Solution: Set $\psi(x) = \dfrac{\varphi(x) - \varphi(-x)}{x}$.

For $x > 0$, we find that $|\psi(x)| = |\frac{1}{x}\int_{-x}^{x} \varphi'(y)dy| \leq 2||\varphi||_{1,0}$.

For $x > 0$, we also find that $|\psi(x)| = |x|^{-1}(1+x^2)^{-1/2}(1+x^2)^{1/2}|\varphi(x) + \varphi(-x)| \leq \frac{1}{x^2}2||\varphi||_{0,1}$.

Via a change of variable, we find $T(\varphi) = \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\infty} \psi(x)\,dx$. Note that ψ is a continuous bounded function, so the limit exists and $T(\varphi) = \int_0^{\infty} \psi(x)\,dx$. Then

$$|T(\varphi)| = \left| \int_0^1 \psi(x)dx + \int_1^{\infty} \psi(x)dx \right| \leq \int_0^1 2||\varphi||_{1,0}dx + \int_1^{\infty} \frac{1}{x^2}2||\varphi||_{0,1}dx = 2||\varphi||_{1,0} + 2||\varphi||_{0,1}.$$

This proves that T has order *at most 1.* Two points were deducted if you omit the "at most" part.

Full credit was awarded without providing a proof that the order cannot be 0. But for completeness, this part of the arguments can be done as follows: For n positive, pick $\varphi_n \in \mathcal{S}$ such that

- $|\varphi_n(x)| \leq 1$ for all x.
- $|\varphi_n(x)| = 0$ for all x such that $|x| \geq 2$.
- $\varphi_n(x) \geq 0$ for $x \geq 0$.
- $\varphi_n(x) \leq 0$ for $x \leq 0$.
- $\varphi_n(x) = 1$ for $x \in [1/n, 1]$.
- $\varphi_n(x) = -1$ for $x \in [-1, -1/n]$.

Observe that then $||\varphi_n||_{0,k} \leq 5^{k/2}$ for every n, but

$$T(\varphi_n) \geq \int_{1/n \leq |x| \leq 1} \frac{1}{x}\varphi(x)\,dx = \int_{1/n \leq |x| \leq 1} \frac{1}{x}\,dx = 2\int_{1/n}^{1} \frac{1}{x}\,dx = 2\log(n) \to \infty.$$

Incidentally, observe that for this sequence, we must necessarily have $||\varphi_n||_{1,0} = ||\varphi_n'||_{\mathrm{u}} \geq n$, since φ_n changes from the value -1 to value 1 in the distance 2/n.

Problem 5: (24p) Consider the Hilbert space $H = L^2(\mathbb{R})$. For this problem, we define H as the closure of the set of all compactly supported smooth functions on $\mathbb{R}$ under the norm

$$||u|| = \Big(\int_{-\infty}^{\infty} |u(x)|^2\,dx\Big)^{1/2}.$$

Which of the following sequences converge weakly in H? Motive your answers briefly.

(a) $(u_n)_{n=1}^{\infty}$ where $u_n(x) = \begin{cases} 1 - |x-n|, & \text{for } x \in [n-1,\, n+1], \\ 0, & \text{for } x \in (-\infty, n-1) \cup (n+1, \infty). \end{cases}$

(b) $(v_n)_{n=1}^{\infty}$ where $v_n(x) = \sin(n\,x)\,e^{-x^2}$.

(c) $(w_n)_{n=1}^{\infty}$ where $w_n(x) = \begin{cases} 1 - |x/n - 1| & \text{for } x \in [0, 2n] \\ 0 & \text{for } x \in (-\infty, 0) \cup (2n, \infty). \end{cases}$

Solution: Let Ω denote the set of smooth functions with compact support. These are by definition dense in H. We use the theorem that says that a sequence is weakly convergent iff it is bounded, and you have weak convergence when measured against any member in a dense set. In the solution, we use Ω as the dense set.

First observe that (u_n) and (v_n) are bounded, but (w_n) is not. We can immediately rule out (w_n).

Fix $f \in \Omega$. Set $M = \sup\{|x| : f(x) \neq 0\}$. Since f has compact support, M is bounded. Now if $n > M + 1$, we find that $(u_n, f) = 0$, so obviously $\lim_{n\to\infty}(u_n, f) = 0$. This shows $u_n \rightharpoonup 0$.

Again fix $f \in \Omega$, and set $g(x) = e^{-x^2} f(x)$. Then as $n \to \infty$,

$$|(v_n, f)| = |\int_{-\infty}^{\infty} \sin(nx) g(x)\,dx| = |-\frac{1}{n}\int_{-\infty}^{\infty} \cos(nx) g'(x)\,dx| \le \frac{1}{n}\int_{-\infty}^{\infty} |g'(x)|\,dx \to 0.$$

We use that g has compact support so the boundary terms in the partial integration vanish, and $\int |g'| < \infty$. This shows $v_n \rightharpoonup 0$.

In summary, (u_n) and (v_n) both converge weakly to zero, but (w_n) does not converge weakly.