Problem 11.22: Set $T = \mathrm{sign}(t)$. We seek to prove that $\check{T} = \alpha \mathrm{PV}(1/x)$ for some α.

For $N = 1, 2, 3, \dots$, set $T_N = \chi_{[-N,N]}\, T$. Then $T_N \to T$ in $\mathcal{S}^*$ since for any $\varphi \in \mathcal{S}$, we have

$$\langle T_n,\, \varphi\rangle = \int_{-N}^{N} \mathrm{sign}(x)\,\varphi(x)\,dx \to \int_{-\infty}^{\infty} \mathrm{sign}(x)\,\varphi(x)\,dx = \langle T,\, \varphi\rangle.$$

Since the Fourier transform is a continuous operator on $\mathcal{S}^*$, we know that $\check{T}$ is the limit of the sequence $(\check{T}_N)_{N=1}^{\infty}$.

Since $T_N \in L^1$, we can compute $\check{T}_N$ by directly evaluating the integral. We find that

$$\check{T}_N(x) = \beta\frac{1 - \cos(N\,x)}{x} \tag{1}$$

for some constant β. If $\varphi \in \mathcal{S}$, then

$$\begin{aligned}\langle\frac{1-\cos(N\,x)}{x},\, \varphi\rangle &= \int_{\mathbb{R}} \frac{1-\cos(N\,x)}{x}\,\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{1-\cos(N\,x)}{x}\,\varphi(x)\,dx\\ &= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{1}{x}\varphi(x)\,dx - \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\cos(N\,x)\,\frac{1}{x}\varphi(x)\,dx\\ &= \langle \mathrm{PV}(1/x),\, \varphi\rangle - \langle\cos(N\,x)\,\mathrm{PV}(1/x),\, \varphi\rangle.\end{aligned}$$

It follows that formula (1) can be written $\check{T}_N(x) = \beta\,\mathrm{PV}(1/x) - \beta\,\cos(N\,x)\,\mathrm{PV}(1/x)$.

It remains to prove that $\cos(N\,x)\,\mathrm{PV}(1/x) \to 0$ in $\mathcal{S}'$. We find that

$$\begin{aligned}\langle\cos(N\,x)\,\mathrm{PV}(1/x),\, \varphi\rangle &= \langle \mathrm{PV}(1/x),\, \cos(N\,x)\,\varphi\rangle\\ &= \int_0^{\infty}\cos(N\,x)\,\frac{1}{x}\,\varphi(x)\,dx + \int_{-\infty}^{0}\cos(N\,x)\,\frac{1}{x}\varphi(x)\,dx\\ &= \int_0^{\infty}\cos(N\,x)\,\frac{\varphi(x)-\varphi(-x)}{x}\,dx.\end{aligned}$$

Now set $\psi(x) = \frac{\varphi(x)-\varphi(-x)}{x}$. Then ψ is a continuously differentiable, quickly decaying function on $[0,\,\infty)$, so we can perform a partial integration to obtain

$$\begin{aligned}\left|\int_0^{\infty}\cos(N\,x)\,\frac{\varphi(x)-\varphi(-x)}{x}\,dx\right| &= \left|\left[\frac{\sin(N\,x)}{N}\,\psi(x)\right]_0^{\infty} - \int_0^{\infty}\frac{\sin(N\,x)}{N}\,\psi'(x)\,dx\right|\\ &\leq \frac{1}{N}\int_0^{\infty}|\psi'(x)|\,dx.\end{aligned}$$

If we can prove that $\int_0^{\infty}|\psi'(x)|\,dx < \infty$, we will be done. First note that for $x \in [0,\,1]$, $\psi(x) = 2\varphi'(0) + O(x^2)$, so for $x \in [0,\,1]$, we have $|\psi'(x)| \leq C_1$ for some finite C_1. For $x \in [1, \infty)$, we have

$$|\psi'(x)| = \left|\frac{\varphi'(x)+\varphi'(-x)}{x} - \frac{\varphi(x)-\varphi(-x)}{x^2}\right| \leq 2\frac{||\varphi||_{1,1}}{x^2} + 2\frac{||\varphi||_{0,0}}{x^2} = \frac{C_2}{x^2}.$$

and so

$$\int_0^{\infty}|\psi'(x)|\,dx \leq \int_0^1 C_1\,dx + \int_1^{\infty}\frac{C_2}{x^2}\,dx < \infty.$$