Homework set 8 — APPM5450, Spring 2014

From the textbook: 11.1(b,c), 11.2, 11.6, 11.7, 11.8, 11.10.

Problem 1: Define a function f on $\mathbb{R}$ via $f(x) = |x|$. Compute carefully the first and the second derivatives of f where the differentiation is understood in a distributional sense.

Problem 2: Prove that if $f \in C^\infty(\mathbb{R}^d)$, and for every $\alpha \in \mathbb{Z}^d$, there exist finite C and N such that $|\partial^\alpha f(x)| \leq C(1 + |x|^N)$, then $f\varphi \in \mathcal{S}$ whenever $\varphi \in \mathcal{S}$. Moreover, prove that if $\varphi_n \to \varphi$ in $\mathcal{S}$, then $f\varphi_n \to f\varphi$ in $\mathcal{S}$.

Problem 3: Demonstrate that a tempered function is not necessarily of at most polynomial growth by constructing a continuous function f on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty, \tag{1}$$

but

$$\sup_{x\in\mathbb{R}} \frac{|f(x)|}{(1 + |x|^2)^{k/2}} = \infty, \qquad \forall\, k \in \{0,\, 1,\, 2,\, \dots\}. \tag{2}$$

If you would like to make the problem slightly harder, then construct a function f that satisfies (2) and also

$$\int_{-\infty}^{\infty} \big(1 + |x|^2\big)^{k/2} |f(x)|\, dx < \infty, \qquad \forall\, k \in \{0,\, 1,\, 2,\, \dots\}. \tag{3}$$

Problem 4 (optional): Let $\mathcal{D}$ denote the linear space $C_{\rm c}^\infty(\mathbb{R}^d)$. We define a topology on $\mathcal{D}$ by saying that $\varphi_n \to \varphi$ if and only if there exists a compact set $K \subseteq \mathbb{R}^d$ such that $\text{supp}(\varphi_n) \subseteq K$ for all n, and $||\partial^\alpha \varphi_n - \partial^\alpha \varphi||_{\rm u} \to 0$ for all $\alpha \in \mathbb{Z}_+^d$.
(a) Prove that $\mathcal{D}$ is a linear subspace of $\mathcal{S}$.
(b) Prove that the set $\mathcal{D}$ is not closed in the topology of $\mathcal{S}$.
(c) Prove that if $\varphi_n \to \varphi$ in $\mathcal{D}$, then $\varphi_n \to \varphi$ in $\mathcal{S}$.