Homework set 3 — Solutions — APPM5450, Spring 2014

Problem 8.6: Suppose that U is an isometric isomorphism. This by definition implies that U is bijective, so all we need to prove is that it preserves the inner product. However, this follows directly from the polarization identity (equation (6.5) in the text), and the fact that the norm is preserved.

Problem 1: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for H. Given a bounded sequence of complex number $(\lambda_n)_{n=1}^{\infty}$, define the operator A by setting

$$A\,u = \sum_{n=1}^{\infty} \lambda_n\,\varphi_n\,\langle \varphi_n,\,u\rangle.$$

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Prove that $A^*u = \sum_{n=1}^{\infty} \bar{\lambda}_n \varphi_n\,\langle\varphi_n,\,u\rangle$. Conclude that A is self-adjoint iff all λ_n's are real. When is A skew-symmetric? When is A non-negative / positive / coercive?

Note: This problem is an entirely cosmetic generalization of the example we did on the black-board with $H = \ell^2(\mathbb{N})$, and

$$A(x_1,\,x_2,\,x_3,\,\dots) = (\lambda_1 x_1,\,\lambda_2 x_2,\,\lambda_3 x_3,\,\dots).$$

Solution: (a) Set $M = \sup_n |\lambda_n|$. Then by Parseval

$$||Au||^2 = \sum_{n=1}^{\infty} |\lambda_n \langle\varphi_n,\,u\rangle|^2 \leq \sum_{n=1}^{\infty} M^2 |\langle\varphi_n,\,u\rangle|^2 = M^2||u||^2.$$

Conversely, suppose $M = \lim_{j\to\infty} |\lambda_{n_j}|$. Then

$$||A|| = \sup_{||u||=1} ||Au|| \geq \limsup_{j\to\infty} ||A\varphi_{n_j}|| = \limsup_{j\to\infty} |\lambda_{n_j}| = M.$$

(b) We find

$$\langle Au, v\rangle = \langle \sum_{n=1}^{\infty} \lambda_n\,\varphi_n\,\langle\varphi_n,\,u\rangle, v\rangle = \sum_{n=1}^{\infty} \overline{\lambda_n \langle\varphi_n,\,u\rangle} \langle\,\varphi_n\,,v\rangle$$

$$= \sum_{n=1}^{\infty} \overline{\lambda_n}\langle u,\varphi_n\rangle\langle\,\varphi_n\,,v\rangle = \langle u, \underbrace{\sum_{n=1}^{\infty} \overline{\lambda_n}\varphi_n\langle\,\varphi_n\,,v\rangle}_{=A^*v}\rangle = \langle u, A^*v\rangle.$$

It follows that

$$\begin{array}{rcl}
A \text{ self-adjoint} & \Leftrightarrow & \bar{\lambda}_j = \lambda_j \quad \forall j \\
A \text{ skew-adjoint} & \Leftrightarrow & \bar{\lambda}_j = -\lambda_j \quad \forall j \\
A \text{ non-negative} & \Leftrightarrow & \lambda_j \in [0,\infty) \quad \forall j \\
A \text{ positive} & \Leftrightarrow & \lambda_j \in (0,\infty) \quad \forall j \\
A \text{ coercive} & \Leftrightarrow & \text{there is a } c > 0 \text{ such that } \lambda_j \in [c,\infty) \quad \forall j
\end{array}$$

Problem 2: Consider the Hilbert space $H = L^2([-\pi,\, \pi])$, and the operator $A \in \mathcal{B}(H)$ defined by $[A\,u](x) = |x|\,u(x)$. Prove that A is self-adjoint and positive, but not coercive. Prove that

$$\langle u,\, v\rangle_A = \langle A\,u,\, v\rangle$$

is an inner product on H, but that the topology generated by (the norm generated by) this inner product is *not* equivalent to the topology generated by the L^2-norm.

Solution: Note that

$$\langle Au,\, u\rangle = \int_{-\pi}^{\pi} \overline{|x|\,u(x)}\,u(x)\,dx = \int_{-\pi}^{\pi} |x|\,|u(x)|^2\,dx \geq 0,$$

which immediate shows that A is non-negative.

To prove that A is positive, suppose that $\langle Au,\, u\rangle = 0$. Then $|x|\,|u(x)|^2 = 0$ for all $x \neq 0$, which is to say, u is the zero function in L^2. (To be rigorous, $\langle Au,\, u\rangle = 0$ implies that $|x|\,|u(x)|^2 = 0$ except for on a "set of measure zero". We will return to this point later in the course once we have covered measure theory.)

Verifying that $\langle\cdot,\,\cdot\rangle_A$ is an inner product is straight-forward.

To prove that the norms $||\cdot||_A$ and $||\cdot||$ are not equivalent, set

$$u_n(x) = \begin{cases} \sqrt{n/2} & \text{when } |x| \leq 1/n \\ 0 & \text{when } |x| > 1/n. \end{cases}$$

It is easily verified that $||u_n|| = 1$, while $||u_n||_A \leq 1/\sqrt{n}$. It follows that

$$\inf_{u\neq 0} \frac{||u||_A}{||u||} \leq \inf_n \frac{||u_n||_A}{||u_n||} \leq \inf_n \frac{1/\sqrt{n}}{1} = 0.$$

Problem 3: Set $H = \ell^2(\mathbb{Z})$ and let R denote the right-shift operator (so that if $y = R\,x$, then $y_n = x_{n-1}$). Construct R^*. Prove that $R\,R^* = R^*\,R = I$, which is to say that R is "unitary." (Is either the right or the left-shift operator on $\ell^2(\mathbb{N})$ unitary?)

Solution: This should be simple.

Problem 4: Consider the Hilbert space $L^2(\mathbb{T})$. Let k denote a continuous function on $\mathbb{T}^2$ that takes on complex values. Let A denote the operator $[Au](x) = \int_{\mathbb{T}} k(x,y)\,u(y)\,dy$. Prove that $[A^*u](x) = \int_{\mathbb{T}} \overline{k(y,x)}\,u(y)\,dy$. Conclude that A is self-adjoint iff $k(x,y) = \overline{k(y,x)}\ \forall\ x,y \in \mathbb{T}$.

Solution: Suppose that $u, v \in \mathcal{P}$ (i.e. they are finite linear combinations of trig functions). Then

$$\langle Au,\, v\rangle = \int_{\mathbb{T}} \overline{\int_{\mathbb{T}} k(x,y)\,u(y)\,dy}\,v(x)\,dx = \int_{\mathbb{T}\times\mathbb{T}} \overline{k(x,y)}\,\overline{u(y)}\,v(x)\,dA = \int_{\mathbb{T}} \overline{u(y)} \underbrace{\int_{\mathbb{T}} \overline{k(x,y)}\,v(x)\,dx}_{=[A^*v](y)}\,dy.$$

Since we have proven the relationship on a dense subset of L^2, and since the inner product is continuous, the relationship must hold on the entire set.