

Solutions for Homework 1 — APPM5450 — Spring 2014

Problem 7.1:

(a) Fix $\delta > 0$. For $x \in [-\delta/2, \delta/2]$ we have $1 + \cos x \geq 1 + \cos \frac{\delta}{2}$ so

$$(1) \quad \frac{1}{c_n} = \int_{\mathbb{T}} (1 + \cos x)^n dx \geq \int_{-\delta/2}^{\delta/2} \left(1 + \cos \frac{\delta}{2}\right)^n dx = \delta \left(1 + \cos \frac{\delta}{2}\right)^n.$$

Analogously, we find that

$$(2) \quad \int_{|x| \geq \delta} c_n (1 + \cos x)^n dx \leq \int_{|x| \geq \delta} c_n (1 + \cos \delta)^n dx \leq c_n 2\pi (1 + \cos \delta)^n.$$

Inserting (1) into (2) and taking the limit, we find (since $1 + \cos \delta < 1 + \cos(\delta/2)$)

$$\lim_{n \rightarrow \infty} \int_{|x| \geq \delta} c_n (1 + \cos x)^n dx \leq \limsup_{n \rightarrow \infty} \frac{2\pi}{\delta} \left(\frac{1 + \cos \delta}{1 + \cos(\delta/2)} \right)^n = 0.$$

(b) See lecture notes.

(c) No, since any function in \mathcal{P} is periodic. Consider for instance $f(x) = x$. Then for any $g \in \mathcal{P}$

$$\|f - g\|_u \geq \max(|f(0) - g(0)|, |f(2\pi) - g(2\pi)|) = \max(|g(0)|, |2\pi - g(0)|) \geq \pi.$$

Problem 7.2: With $e_n(x) = e^{inx}/\sqrt{2\pi}$ we set

$$f_N(x) = \sum_{n=-N}^N \alpha_n e_n(x), \quad \alpha_n = (e_n, f).$$

Set $\beta = 1/\sqrt{2\pi}$. Then

$$f_N(x) = \sum_{n=-N}^N \int_{-\pi}^{\pi} \beta e^{-iny} f(y) dy \beta e^{inx} = \int_{-\pi}^{\pi} \beta^2 \underbrace{\sum_{n=-N}^N e^{in(x-y)}}_{=: D_N(x-y)} f(y) dy.$$

We will next simplify the kernel D_N . To this end, set $\alpha = e^{ix}$. Then

$$D_N = \beta^2 \sum_{n=-N}^N \alpha^n.$$

Moreover,

$$\alpha D_N = \beta^2 \sum_{n=-N}^N \alpha^{n+1}.$$

In consequence,

$$(1 - \alpha) D_N = \beta^2 (\alpha^{-N} - \alpha^{N+1}).$$

It follows that

$$D_N = \beta^2 \frac{\alpha^{-N} - \alpha^{N+1}}{1 - \alpha} = \beta^2 \frac{\alpha^{-(N+1/2)} - \alpha^{N+1/2}}{\alpha^{-1/2} - \alpha^{1/2}} = \frac{1}{2\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

This proves part (a).

Next we set

$$g_N = \frac{1}{N+1} \sum_{n=0}^N f_0 = \frac{1}{N+1} \sum_{n=0}^N D_n * f = \underbrace{\left(\frac{1}{N+1} \sum_{n=0}^N D_n \right)}_{=: F_N} * f.$$

It remains to simplify F_N . We have

$$\begin{aligned} F_N &= \frac{1}{N+1} \sum_{n=0}^N D_n = \frac{1}{N+1} \sum_{n=0}^N \beta^2 \frac{\alpha^{-n} - \alpha^{n+1}}{1 - \alpha} = \frac{\beta^2}{(N+1)(1-\alpha)} \left[\frac{(1/\alpha)^{N+1} - 1}{1/\alpha - 1} - \frac{\alpha^{N+2} - \alpha}{\alpha - 1} \right] \\ &= \frac{\beta^2 \alpha}{(N+1)(1-\alpha)^2} \left[\alpha^{-(N+1)} - 2 + \alpha^{N+1} \right] = \frac{\beta^2 \alpha}{(N+1)(\alpha^{-1/2} - \alpha^{1/2})^2} \left[\alpha^{-(N+1)/2} - \alpha^{(N+1)/2} \right]^2 \\ &= \frac{\beta^2 \alpha}{(N+1)(-2i \sin(x/2))^2} \left[-2i \sin \frac{(N+1)x}{2} \right]^2 = \frac{\beta^2 \alpha}{(N+1)(\sin(x/2))^2} \left[\sin \frac{(N+1)x}{2} \right]^2. \end{aligned}$$

This proves part (b).

For (c), we observe that D_N takes on non-negative values, so it is not an approximate identity. Convolution by D_N provides the best approximation in the L^2 -norm, but it does not guarantee convergence in the uniform norm. In contrast, convolution by F_N does provide convergence in the uniform norm as long as $f \in C(\mathbb{T})$.

Problem 7.3: Start by proving that the two putative bases are in fact orthonormal sets. Then it remains to prove that their closures span the set.

Fix an $f \in L^2(J)$ with $J = [0, \pi]$. To construct a sequence f_N such that $\|f - f_N\|_{L^2(J)} \rightarrow 0$, extend f to the function

$$\bar{f}(x) = \begin{cases} f(x) & x \geq 0, \\ -f(-x) & x < 0. \end{cases}$$

Then let f_N be the standard Fourier series of \bar{f} . Prove that the terms in this series are all sine functions. Since the exponentials form a basis, we know that $\|\bar{f} - f_N\|_{L^2(J)} \rightarrow 0$ where $I = [-\pi, \pi]$. Since $\|\bar{f} - f_N\|_{L^2(J)} = \sqrt{2}\|f - f_N\|_{L^2(J)}$, we then find that $f_N \rightarrow f$ in $L^2(J)$.

To prove that the cosines form a basis, repeat the argument, but do it with the symmetric continuation of f instead of the anti-symmetric one. In other words, set

$$\tilde{f}(x) = \begin{cases} f(x) & x \geq 0, \\ f(-x) & x < 0, \end{cases}$$

and then use that the Fourier series for \tilde{f} involves only cosines.

Problem 7.4: This is a straight-forward calculation. You may want to look the correct answer up in a table to make sure you got the answer right.

Problem 7.5: The argument for the case $d = 1$ was done in class (see posted lecture notes). This argument can easily be modified to the case of d dimensions. Let f_N denote the partial Fourier sum. We need to prove that (f_N) is Cauchy with respect to the uniform norm. If $M < N$, we find

$$\begin{aligned}
 |f_M(x) - f_N(x)| &= \left| \sum_{M < |n| \leq N} \alpha_n e_n(x) \right| \leq \sum_{M < |n| \leq N} |\alpha_n| \\
 &\leq \left(\sum_{M < |n| \leq N} |n|^{-2k} \right)^{1/2} \left(\sum_{M < |n| \leq N} |n|^{2k} |\alpha_n|^2 \right)^{1/2} \\
 &\sim \left(\int_{M \leq |x| \leq N} |x|^{-2k} dx \right)^{1/2} \|f\|_{H^k} \sim \left(\int_M^N r^{-2k} r^{d-1} dr \right)^{1/2} \|f\|_{H^k} \\
 &\leq \left(\int_M^\infty r^{-2k} r^{d-1} dr \right)^{1/2} \|f\|_{H^k} = \frac{1}{\sqrt{2k-d} M^{k-d/2}} \|f\|_{H^k}.
 \end{aligned}$$

Problem 1: Suppose that H is a Hilbert space, and that $(\psi_n)_{n=1}^\infty$ is an ON-set in H . Let \mathcal{P} denote the set of finite linear combinations of elements in $(\psi_n)_{n=1}^\infty$. Prove that:

$$(\psi_n)_{n=1}^\infty \text{ is a basis for } H \iff \mathcal{P} \text{ is dense in } H.$$

Solution: Suppose first that $(\psi_n)_{n=1}^\infty$ is a basis. Given any $f \in H$, define its partial expansion in (ψ_n) as usual:

$$(3) \quad f_N = \sum_{n=1}^N (\psi_n, f) \psi_n$$

Since (ψ_n) is a basis, we know that $f_N \rightarrow f$ in norm. Since $f_N \in \mathcal{P}$, this proves that any function can be approximated arbitrarily well by functions in \mathcal{P} .

Suppose next that \mathcal{P} is dense. Fix an $f \in H$, and define its partial expansion f_N as in (3). We need to prove that $f_N \rightarrow f$. Fix any $\varepsilon > 0$. Since \mathcal{P} is dense, there is a $g \in \mathcal{P}$ such that $\|f - g\| < \varepsilon$. Let N be a number such that $g \in \text{Span}(\psi_1, \psi_2, \dots, \psi_N) =: \mathcal{P}_N$. Now suppose that $M \geq N$. Then since $g \in \mathcal{P}_M$, and f_M is the best possible approximant within \mathcal{P}_M , we find

$$\|f - f_M\| \leq \|f - g\| < \varepsilon.$$

This shows that $f_N \rightarrow f$.

Problem 2: Suppose that $f, g \in C(\mathbb{T})$. Prove that:

(a) $f * g \in C(\mathbb{T})$.

(b) $f * g = g * f$.

Solution:

(a) Set $h = f * g$. That h is periodic follows directly from the periodicity of f :

$$h(x + 2\pi) = \int_{\mathbb{T}} f(x + 2\pi - y)g(y)dy = \int_{\mathbb{T}} f(x - y)g(y)dy = h(x).$$

Next we prove continuity. Fix $\varepsilon > 0$. Since f is uniformly continuous, there is a $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon/(2\pi\|g\|)$ whenever $|x - x'| < \delta$. Now suppose that $|x - x'| < \delta$. Then

$$|h(x) - h(x')| = \left| \int_{\mathbb{T}} (f(x-y) - f(x'-y))g(y)dy \right| \leq \int_{\mathbb{T}} |f(x-y) - f(x'-y)| |g(y)| dy \leq \int_{\mathbb{T}} \frac{\varepsilon}{2\pi\|g\|} \|g\| dy = \varepsilon.$$

(b) Simply use the change of variables $z = x - y$ in the integral. You need to verify that the limits and the minus signs work out as they should, but that should not be hard.