Weak differentiation on $L^2(\mathbb{T})$ (without Fourier methods)

We consider the space $X = L^2(\mathbb{T})$, with the usual norm

$$||f|| = \left(\int_I |f(x)|^2 \, dx \right)^{1/2}.$$

Let Ω denote the set of continuously differentiable functions on $\mathbb{T}$. Note that Ω is dense[1] in X.

Fix a function $f \in X$ and define for $g \in \Omega$ the functional

$$T_f(g) = - \int_{\mathbb{T}} \overline{f(x)} \, g'(x) \, dx.$$

Suppose that there exists a number C (that depends on f) such that

$$|T_f(g)| \le C \, ||g||, \quad \forall \, g \in \Omega.$$

Then T_f is a continuous functional defined on a dense set. It follows that T_f has a unique extension $\tilde{T}_f \in X^*$. By the Riesz representation theorem, we know that there exists a unique $h \in X$ such that

$$\tilde{T}_f(g) = \langle h, g \rangle, \quad \forall \, g \in X.$$

We define this function h to be the weak derivative of f.

Remark 1: If f is a classically differentiable function, then our definition of a weak derivative coincides with the classical definition. To see this, note that if $f \in \Omega$, then using integration by parts, we obtain

$$T_f(g) = - \int_{\mathbb{T}} \overline{f(x)} \, g'(x) \, dx = \int_{\mathbb{T}} \overline{f'(x)} \, g(x) \, dx = \langle f', g \rangle.$$

It follows that in this case

$$\langle f', g \rangle = \langle h, g \rangle \quad \forall \, g \in \Omega,$$

and since Ω is dense in X, we must have $f' = h$.

Remark 2: The definition of a weak derivative given here coincides with the Fourier definition. To see this, note that if $g \in \Omega$, and $f = \sum \alpha_n e_n$ and $g = \sum \beta_n e_n$, then

$$T_f(g) = -\langle f, g' \rangle = - \sum_{n \in \mathbb{Z}} \overline{\alpha_n} \, in\beta_n = \sum_{n \in \mathbb{Z}} \overline{in\alpha_n} \, \beta_n.$$

Since Ω is dense in X, it follows that the number

$$C = \sup_{g \in \Omega} |T_f(g)|$$

is finite if and only if $(in\alpha_n)_{n=-\infty}^{\infty} \in L^2(\mathbb{Z})$, and if it is, then necessarily $h = \sum in\alpha_n e_n$.

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[1]To see that Ω is dense in X, note that Ω contains the set $\mathcal{P}$ of all polynomials on I, and $\mathcal{P}$ is dense in X.