APPM5450 — Applied Analysis: Section exam 3
8:30 – 9:50, April 24, 2013. Closed books.

Problem 1: (36p) No motivation required.

(a) (6p) Define $T \in \mathcal{S}^*(\mathbb{R}^d)$ via $\langle T, \varphi\rangle = \varphi(0)$. What is $\hat{T}$?

(b) (6p) Define $T \in \mathcal{S}^*(\mathbb{R}^d)$ via $\langle T, \varphi\rangle = \varphi(0)$. State for which $s \in \mathbb{R}$ (if any) we have

$$\int_{\mathbb{R}^d} \left(1 + |t|^2\right)^s |\hat{T}(t)|^2\, dt < \infty.$$

(Recall that this is the definition for when $T \in H^s(\mathbb{R}^d)$.)

(c) (6p) Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \chi_{[0,1]}(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$. What is $\hat{T}$?

(d) (6p) Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. What is $\hat{T}$?

(e) (6p) Give an example of a non-zero Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}\varphi = -i\,\varphi$, where i is the imaginary unit.

(f) (6p) Which of the following statements are true:
(1) $\mathcal{F}(L^2(\mathbb{R})) \subseteq C_0(\mathbb{R})$
(2) $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}^*(\mathbb{R})$
(3) $L^p(\mathbb{R}) \subseteq \mathcal{S}^*(\mathbb{R})$ for every $p \in [1,\infty]$.
(4) $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$ when $s > d/2$.
(5) $C^1(\mathbb{R}) \subseteq H^2(\mathbb{R})$.

Problem 2: (24p) Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n\,\cos(nx) & \text{for } |x| \le R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R, if any, is it the case that $f_n \to 0$ in $\mathcal{S}^*(\mathbb{R})$? Please motivate your answer.

Problem 3: (20p)

(a) (5p) State the definition of a σ-algebra.

(b) (5p) State the definition of a measure.

(c) (10p) Let $(X, \mathcal{A}, \mu)$ denote a measure space. Suppose that $\Omega_1,\, \Omega_2 \in \mathcal{A}$. Prove directly from the axioms that $\mu(\Omega_1 \cup \Omega_2) \le \mu(\Omega_1) + \mu(\Omega_2)$. Give a condition for when equality occurs.

Problem 4: (20p) Set $X = [1,\infty)$, let $\mathcal{A}$ denote the power set on X, let $\mathbb{N} = \{0,\,1,\,2,\,3,\,\dots\}$ denote the natural numbers, and define a measure μ on $\mathcal{A}$ via

$$\mu(\Omega) = \sum_{j \in \Omega \cap \mathbb{N}} 2^{-j}.$$

Which of the following functions are Lebesgue integrable with respect to $(X, \mathcal{A}, \mu)$? For the functions that are Lebesgue integrable, state the value of $\int_X f\, d\mu$. Please motivate briefly.

(a) $f_1(x) = e^x$

(b) $f_2(x) = e^{-x}$

(c) $f_3(x) = e^x - 9\,e^{-x}$

(d) $f_4(x) = e^x\, \cos(\pi x)$