APPM5450 — Applied Analysis: Section exam 2
8:30 – 9:50, March 20, 2013. Closed books.

Problem 1: (5p) Let A be a compact self-adjoint operator on a Hilbert space H. You know that $\sigma(A) = \{0\} \cup \{1,\, 1/2,\, 1/3,\, 1/4,\, \dots\}$. Set $\lambda = 1/\pi$. What is $||(A - \lambda I)^{-1}||$? No motivation required.

Problem 2: (5p) Let H be a Hilbert space, and suppose that $A \in \mathcal{B}(H)$. You know that $||A|| = 2$ and $A^3 = 0$. What can you say about $\sigma(A)$? Please motivate briefly.

Problem 3: (10p) Set $H = L^2(\mathbb{R})$, and $[Au](x) = u(x-1)$. No motivation required.

(a) (4p) Specify $\inf\limits_{\lambda \in \sigma(A)} |\lambda|$.

(b) (4p) Specify $\sup\limits_{\lambda \in \sigma(A)} |\lambda|$.

(c) (2p) Specify $\sigma(A)$. *Hint: This problem could be slightly challenging and is only worth 2p!*

Problem 4: (30p) Consider the Hilbert space $H = L^2(\mathbb{R})$. Define a function $f \in C_{\mathrm{b}}(\mathbb{R})$ via

$$f(x) = \begin{cases} \arctan(x) & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Define a linear operator $A :\ H \rightarrow H$ via

$$[Au](x) = f(x)\, u(x).$$

(a) (5p) Specify $||A||$. No motivation required.

(b) (5p) Is A self-adjoint? Compact? Non-negative? Positive? No motivation required.

(c) (12p) Specify $\sigma(A)$, $\sigma_{\mathrm{r}}(A)$, $\sigma_{\mathrm{c}}(A)$, and $\sigma_{\mathrm{p}}(A)$. No motivation required.

(d) (8p) Provide brief motivations for your claims in (c).

Problem 5: (25p) Consider four maps $T_i\, :\, \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ defined via

$$\begin{aligned} T_1(\varphi) =& \sup_{x \in \mathbb{R}} \varphi(x) \\ T_2(\varphi) =& \varphi'(2) \\ T_3(\varphi) =& \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \varphi(x)\, dx \\ T_4(\varphi) =& \lim_{\varepsilon \rightarrow 0} \left[\int_{|x| \geq \varepsilon} \frac{1}{x^2} \varphi(x)\, dx - \frac{2\varphi(0)}{\varepsilon} \right]. \end{aligned}$$

No motivation required for these problems:

(a) (11p) Which of the maps T_i belong to $\mathcal{S}^*(\mathbb{R})$?

(b) (7p) For the maps that do belong to $\mathcal{S}^*(\mathbb{R})$, state the *order* of the map.

(c) (7p) For the maps that do belong to $\mathcal{S}^*(\mathbb{R})$, specify a continuous function f_i on $\mathbb{R}$ such that T_i is a derivative (first, second, third, ...) of f_i.

Problem 6: (25p) Let φ be a Schwartz function on $\mathbb{R}$. Define for $N = 1, 2, 3, \ldots,$ the function ψ_N via

$$\psi_N(x) = \sum_{n=1}^{N} \frac{1}{n^2} \varphi(x - n).$$

Please motivate your answers to the following questions *briefly.*

(a) (7p) Is the function ψ_N necessarily a Schwartz function?

(b) (9p) Does the sequence $(\psi_N)_{N=1}^{\infty}$ necessarily converge in $C_{\mathrm{b}}(\mathbb{R})$?

(c) (9p) Does the sequence $(\psi_N)_{N=1}^{\infty}$ necessarily converge in $\mathcal{S}(\mathbb{R})$?