

Homework set 12 — APPM5450, Spring 2013

From the textbook: 12.2, 12.3, 12.5, 12.7.

Problem: Let (X, μ) be a measure space and consider the space $L^\infty(X, \mu)$ consisting of all measurable functions from X to \mathbb{R} such that

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)| < \infty.$$

Prove that $L^\infty(X, \mu)$ is closed under the norm $\|\cdot\|_\infty$.

Hint: You may want to start as follows:

- (1) Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(X, \mu)$.
- (2) For each positive integer k , there exists and N_k such that for $m, n \geq N_k$, $\|f_n - f_m\|_\infty < 1/k$.
- (3) For each k , and for each $m, n \geq N_k$, let Ω_{mn}^k denote the set of all $x \in X$ such that $|f_m(x) - f_n(x)| < 1/k$. What can you tell about Ω_{mn}^k in light of (2)?
- (4) Set $\Omega^k = \bigcap_{m, n = N_k}^\infty \Omega_{mn}^k$. What do you know about Ω^k in view of your conclusion from (3)?
- (5) Set $\Omega = \bigcap_{k=1}^\infty \Omega^k$. What do you know about Ω in view of your conclusion from (4)?
- (6) What can you tell about $(f_n(x))_{n=1}^\infty$ for $x \in \Omega$?