Homework 10

11.18) Prove that if $g \in L^2$ satisfies $g(-x) = \overline{g(x)}$, then $\hat{g}$ is real-valued.

Set $C = \dfrac{1}{(2\pi)^{n/2}}$. Then

$$\overline{\hat{g}(k)} = \overline{C\int_{-\infty}^{\infty} g(x)e^{-ikx}dx} = C\int_{-\infty}^{\infty} \overline{g(x)}e^{ikx}dx \overset{given}{=} C\int_{-\infty}^{\infty} g(-x)e^{ikx}dx \overset{y=-x}{=} C\int_{-\infty}^{\infty} g(y)e^{-iky}dx = \hat{g}(k)$$

Note that the equality denoted by "given" uses $g(-x) = \overline{g(x)}$ and the equality denoted by "$y = -x$" is a substitution.

11.13)

(a) Prove the following equations for the Fourier transform of translates and convolutions.
(b) Prove the corresponding results for the derivatives and translates of tempered distributions.
(c) Prove the corresponding results for the convolution of a test function with a tempered distribution.

For $\phi,\psi \in S$ and $h \in R^n$

(1) $F[\tau_h \phi] = e^{-ik\cdot h}\hat{\phi}$

(2) $F[e^{ix\cdot h}\phi] = \tau_h \hat{\phi}$

(3) $F[\phi * \psi] = (2\pi)^{n/2}\hat{\phi}\hat{\psi}$

Some of these were covered in class so only selected ones will be shown here.

Assume $\phi \in S$ is a test function (definition on page 288). We will prove (1) and (2).

Recall the definition of the Fourier transform $\hat{g}(k) = \underbrace{\left(\frac{1}{(2\pi)^{2}}\right)^{d}}_{=\beta^d} \int_{-\infty}^{\infty} g(x) e^{-ikx}\,dx$

(1) $$F[\tau_h \phi](k) = \beta^d \int_{-\infty}^{\infty} \phi(x-h) e^{-ikx}\,dx \overset{x=y+h}{=} \beta^d \int_{-\infty}^{\infty} \phi(y) e^{-ik(y+h)}\,dx = \beta^d \int_{-\infty}^{\infty} \phi(y) e^{-iky} e^{-ikh}\,dx = e^{-ikh}\hat{\phi}$$

(2) $$F[e^{ix\cdot h}\phi](k) = \beta^d \int_{-\infty}^{\infty} e^{ix\cdot h}\phi(x) e^{-ikx}\,dx = \beta^d \int_{-\infty}^{\infty} \phi(x) e^{-ix(k-h)}\,dx = \hat{\phi}(k-h) = \tau_h \hat{\phi}(k)$$

Assume $T \in S^*$ is a tempered distribution (definition on page 291). We will prove (1) and (2).
Recall the definition of the Fourier transform $\hat{T} = F[T]$ where $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$

(1') $$\langle F[\tau_h T], \phi \rangle = \langle T, \tau_{-h}\hat{\phi} \rangle \overset{above}{=} \langle T, F[e^{-ikh}\phi] \rangle = \langle F[T], e^{-ikh}\phi \rangle = \langle e^{-ikh}\hat{T}, \phi \rangle \Rightarrow F[\tau_h T] = e^{-ikh}\hat{T}$$

(2') $$\langle F[e^{ikh}T], \phi \rangle = \langle e^{ikh}T, \hat{\phi} \rangle = \langle T, e^{ikh}\hat{\phi} \rangle \overset{above}{=} \langle T, F[\tau_{-h}\phi] \rangle = \langle \hat{T}, \tau_{-h}\phi \rangle = \langle \tau_h \hat{T}, \phi \rangle \Rightarrow F[e^{ikh}T] = \tau_h \hat{T}$$

11.15) Let $f_n : R \to R$ be the function below. Show that the sequence (f_n) converges in $S^*(R)$ as $n \to \infty$, and determine its distributional limit.

$$f_n(x) = \begin{cases} n^2 & -1/n < x < 0 \\ -n^2 & 0 < x < 1/n \\ 0 & else \end{cases}$$

Note that $F_n(x) = \int_{-\infty}^{x} f_n(y)dy = \begin{cases} n^2(x+1/n) & -1/n < x < 0 \\ -n^2(-x+1/n) & 0 < x < 1/n \\ 0 & else \end{cases}$

Fix $\phi \in S$. Then

$$\langle f_n, \phi \rangle = \langle F_n', \phi \rangle = -\langle F_n, \phi' \rangle = -\int_{-\infty}^{\infty} F_n(x)\phi'(x)dx = -\int_{-\infty}^{\infty} \underbrace{F_n(x)}_{\int F_n = 1} \phi'(0)dx - \int_{-\infty}^{\infty} F_n(x)(\phi'(x) - \phi'(0))dx$$

With a little rearranging we have

$$\left|\langle f_n, \phi \rangle - \phi'(0)\right| = \left| \int_{-\infty}^{\infty} F_n(x)(\phi'(x) - \phi'(0))dx \right| = \int_{-1/n}^{1/n} \underbrace{F_n(x)}_{\int F_n = 1} \underbrace{\left|\phi'(x) - \phi'(0)\right|}_{\le |x| \cdot \|\phi''\|_u \le \frac{1}{n}\|\phi''\|_u} dx \le \frac{1}{n}\|\phi''\|_u \xrightarrow{n\to\infty} 0$$

So $\langle f_n, \phi \rangle \to -\phi'(0) = \langle \delta, -\phi' \rangle = \langle \delta', \phi \rangle \Rightarrow f_n \to \delta'$ in $S^*(R)$.

11.16) Let $f \in L^1(R^3)$ be a rotationally invariant function in the sense that there is a function $g: R^+ \to C$ s.t. $f(x) = g(|x|)$. Prove that the Fourier transform of f is a continuous function $\hat{f}$ that is also rotation invariant, and $\hat{f}(t) = h(|t|)$, where $h(t) = \frac{1}{t}\sqrt{\frac{2}{\pi}}\int_0^\infty r\sin(tr)g(r)dr$.

We will convert to polar coordinates where $\tau = |t|, r = |x|$, and B_r is the surface of the sphere. Note that this means that the system is such that t aligns with the x_3 (or z) axis.

$$\hat{f}(t) = \left(\frac{1}{2\pi}\right)^{3/2}\int_{R^3} e^{-ix\cdot t} f(x)dx = \left(\frac{1}{2\pi}\right)^{3/2}\int_0^\infty\int_{B_r} e^{-ix\cdot t} g(r)dAdr = (1)$$

Let's pull out one part and simplify it down

$$\int_{B_r} e^{-ix\cdot t}dA = \int_0^{2\pi}\int_0^{\pi} e^{-ir\tau\cos(\theta)}r^2\sin(\theta)d\theta d\phi = r^2 2\pi\int_0^{\pi} e^{-ir\tau\cos(\theta)}\sin(\theta)d\theta = r^2 2\pi\left[\frac{e^{-ir\tau\cos(\theta)}}{ir\tau}\right]_0^{\pi} =$$

$$= \frac{r2\pi}{\tau}\underbrace{\left(e^{ir\tau} - e^{-ir\tau}\right)}_{=2i\sin(r\tau)} = \frac{4r\pi}{\tau}\sin(r\tau)$$

We can now continue from (1)

$$\hat{f}(t) = \cdots = (1) = \left(\frac{1}{2\pi}\right)^{3/2}\frac{4\pi}{\tau}\int_0^\infty r\sin(r\tau)g(r)dr = \frac{1}{\tau}\sqrt{\frac{2}{\pi}}\int_0^\infty r\sin(\tau r)g(r)dr$$