Homework set 10 — APPM5450, Spring 2013

From the textbook: 11.18, 11.13, 11.16.

In 11.16, you're free to assume that f is smooth (or that $f \in \mathcal{S}(\mathbb{R}^3)$), if you like. You may also assume that $f \in L^1$ in 11.18, but please return to the problem once we've described the action of $\mathcal{F}$ on L^2.

Problem 1: Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n\,\cos(nx) & \text{for } |x| \le R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R, if any, is it the case that $f_n \to 0$ in $\mathcal{S}^*$?

Problem 2: (Optional review of old material.) Prove that $C_{\mathrm{c}}(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. Prove that $C_0(\mathbb{R}^d)$ is a closed subset of $C_{\mathrm{b}}(\mathbb{R}^d)$.

What follows is a set of review questions for Chapter 11. They are not part of the home work but you may find them useful in preparing for the third midterm and the final:

What does it means for $\varphi_n \to \varphi$ in $\mathcal{S}$?

Make absolutely sure that you understand problems like 11.4, 11.10a.

Prove that if $\varphi_n \to \varphi$ in $\mathcal{S}$, then $x\varphi_n(x) \to x\varphi(x)$ and $\partial\varphi_n \to \partial\varphi$ in $\mathcal{S}$.

Let T be a linear map from $\mathcal{S}$ to $\mathbb{R}$. What does it mean for T to be continuous? Prove that if there exists a finite C and a finite N such that $|T(\varphi)| \le C \sum_{|\alpha|,n \le N} ||\varphi||_{\alpha,n}$, then T is continuous.

Let $T \in \mathcal{S}^*(\mathbb{R}^d)$, and let α be a multi-index. Define $x^\alpha T$. Prove that what you define is a tempered distribution.

Prove that $n^2 \sin(nx) \to 0$ in $\mathcal{S}^*$.

Is the Schwartz space dense in $\mathcal{S}^*$?

Prove that $\sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty \;\forall \alpha, \beta \quad \Leftrightarrow \quad \sup_x |(1 + |x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty \;\forall \alpha, k$.

Assume that $\int |f|^2 < \infty$, set $\langle T,\, \varphi\rangle = \int f\,\varphi$. Prove that $T \in \mathcal{S}^*$.

Let H be a function such that $H(x) = 1$ if $x \ge 0$, zero otherwise. Prove that $H \in \mathcal{S}^*$. Calculate H'. Let H_R denote the function that is 1 when $0 \le x \le R$ and zero otherwise. Prove that $H_R \to H$ in $\mathcal{S}^*$ as $R \to \infty$.

Let ψ be a Schwartz function such that $\int \psi = 0$. Set $\varphi_n(x) = n\,\psi(n\,x)$. Does φ_n converge in $\mathcal{S}$? Does φ_n converge in $\mathcal{S}^*$?

Prove that $\mathrm{PV}(1/x)$ is a continuous functional on $\mathcal{S}$.

What is the distributional derivative of $\mathrm{PV}(1/x)$?

Define $\hat{T}$ for $T \in \mathcal{S}^*$. Prove that what you define is a continuous map on $\mathcal{S}$.