

9.10 (slightly reworded): Consider the Hilbert space $H = l^2(\mathbb{N})$ and the operators L and R defined by

$$\begin{aligned} R(x_1, x_2, x_3, \dots) &= (0, x_1, x_2, \dots), \\ L(x_1, x_2, x_3, \dots) &= (x_2, x_3, x_4, \dots). \end{aligned}$$

Determine and classify the spectra of L and R .

Solution: First we determine the eigenvalues. Let λ be a complex number and consider the equation $Rx = \lambda x$. It is easily seen that the only solution is $x = 0$ (ones needs to distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$), and so $\sigma_p(R) = \emptyset$. Next consider the equation $Lx = \lambda x$. The only solution is $x = x_1(1, \lambda, \lambda^2, \dots)$, so we can construct eigenvectors in H if and only if $|\lambda| < 1$. Thus $\sigma_p(L) = \{\lambda : |\lambda| < 1\}$.

Consider λ such that $|\lambda| > 1$: Since $\|L\| = \|R\| = 1$, we immediately find that $\lambda \notin \sigma(L)$ and $\lambda \notin \sigma(R)$.

Consider λ such that $|\lambda| < 1$: We already determined that $\lambda \in \sigma_p(L)$. Since $R = L^*$, this fact gives us information about the spectrum of R as well since

$$\overline{\text{ran}(R - \lambda I)} = (\ker(R^* - \bar{\lambda}I))^\perp = (\ker(L - \bar{\lambda}I))^\perp \neq \{0\}.$$

It follows that $\lambda \in \sigma(R)$, but that $\lambda \notin \sigma_c(R)$. Moreover, we determined previously that $\sigma_p(R) = \emptyset$, so we must have that $\lambda \in \sigma_r(R)$.

Consider λ such that $|\lambda| = 1$: First note that since the spectrum of any operator is closed, and the open unit disc belongs to the spectra of both L and R , we know that λ belongs to both $\sigma(L)$ and $\sigma(R)$. We also know that λ is not an eigenvalue of either L or R . Suppose that λ is in the residual spectrum of L . It would then follow that $\bar{\lambda} \in \sigma_p(L^*) = \sigma_p(R)$, but we know that this is not the case. Thus $\lambda \in \sigma_c(L)$. The proof that $\lambda \in \sigma_c(R)$ is analogous.