Applied Analysis (APPM 5450): Midterm 2 — Solutions

8.30am – 9.50am, Mar. 14, 2011. Closed books.

The problems are worth 20 points each. Briefly motivate all answers except those to Problem 1.

Problem 1: No motivation required for these questions.

(a) Give an example of a bounded linear operator on a Hilbert space that is positive, but not coercive.

(b) Let H be an infinite dimensional Hilbert space. Which of the following sets can be the spectrum of a compact self-adjoint operator?
(1) $A_1 = \{1/n\}_{n=1}^{\infty} = \{1,\, 1/2,\, 1/3,\, 1/4,\, \dots\}$
(2) $A_2 = \{1\} \cup \{1 - 1/n\}_{n=1}^{\infty} = \{1,\, 0,\, 1/2,\, 2/3,\, 3/4,\, 4/5,\, \dots\}$.
(3) $A_3 = \{0,\, 1\} \cup \{e^{i/n}\}_{n=1}^{\infty} = \{0,\, 1,\, e^{i},\, e^{i/2},\, e^{i/3},\, e^{i/4},\, \dots\}$.
(4) $A_4 = \{1,\, 2,\, 3\}$.
(5) $A_5 = \{-1,\, 0\}$.

(c) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}$. What is $\langle \delta'',\, \varphi \rangle$?

(d) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}$. What is $\delta'' * \varphi$?

Solution:

(a) There are obviously many possible examples. A couple of simple ones:
- $H = L^2(I)$ where $I = [0,1]$ and $[A\,u](x) = x\,u(x)$.
- $H = \ell^2(\mathbb{N})$ and $A\,(x_1,\, x_2,\, x_3,\, x_4,\, \dots) = (\frac{1}{1}\,x_1,\, \frac{1}{2}\,x_2,\, \frac{1}{3}\,x_3,\, \frac{1}{4}\,x_4,\, \dots)$

(b) Only A_5. (Grading guide: -2p for each mistake.)

(a) A_1 does not include zero (and is also not closed).
(b) A_2 has an accumulation point at 1.
(c) A_3 is not a subset of the real line.
(d) A_4 does not include zero.
(e) Let u be a non-zero vector in H and set $A\,x = -\frac{1}{||u||^2}\,(u,\, x)\,u$.
Then A is self-adjoint and compact, and $\sigma(A) = A_5$.

(c) -2

(d) The <u>function</u> $x \mapsto \varphi''(x) = (4\,x^2 - 2)\,e^{-x^2}$.

Problem 2: Set $H = \ell^2(\mathbb{Z})$ and let $A \in \mathcal{B}(H)$ denote the rightshift operator (i.e. if $u \in H$ and $v = A\,u$, then $v_n = u_{n-1}$).

(a) Let λ be a complex number such that $|\lambda| = 1$. Prove that you can construct $u^{(n)} \in H$ such that $||u^{(n)}|| = 1$ and $\lim_{n\to\infty} ||A\,u^{(n)} - \lambda\,u^{(n)}|| = 0$.

(b) Determine the spectrum of A.

Solution:

(a) Suppose $|\lambda| = 1$ and set $R_\lambda = A - \lambda\,I$. First verify that R_λ is injective by noting that if $R_\lambda u = 0$, then $u_n = \lambda^{-n}\,u_0$ which implies that $|u_n| = |u_0|$ for all n. The only solution is therefore $u = 0$. Next observe that the range of R_λ is dense since

$$\overline{\mathrm{ran}(A - \lambda\,I)} = \big(\ker(A^* - \lambda\,I)\big)^\perp = \{0\}^\perp = H.$$

(The proof that $A - \lambda I$ is injective immediately carries over to a proof that $A^* - \lambda I$ is injective since A^* is simply left-shift.) Finally observe that $A - \lambda I$ is not onto since, e.g., the zero'th canonical basis vector $e^{(0)}$ does not belong to the range.[1] The closed range theorem now implies that R_λ cannot be coercive since its range is not closed.

(b) Set

$$D = \{\lambda \in \mathbb{C} : \ |\lambda| = 1\}.$$

We proved in part (a) that $D \subseteq \sigma_{\mathrm{c}}(A)$. Observe next that A is a unitary operator. It follows[2] that $\sigma(A) \subseteq D$ and consequently

$$\sigma(A) = \sigma_{\mathrm{c}}(A) = D \qquad \sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{r}}(A) = \emptyset.$$

Alternative explicit proof: Let $\mathcal{F} :\ L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$ denote the standard Fourier transform. We will exploit that $\mathcal{F}$ is unitary, and consequently the operator $T = \mathcal{F}^*\,A\,\mathcal{F}$ has the same spectral properties as A. A simple calculation shows that

$$[T\,U](x) = e^{ix}\,U(x).$$

Given a λ such that $|\lambda| = 1$, pick θ such that $\lambda = e^{i\theta}$. Then set

$$U^{(n)}(x) = \begin{cases} \sqrt{\frac{n}{2}} & \text{when } |x - \theta| \leq 1/n \\ 0 & \text{when } |x - \theta| > 1/n. \end{cases}$$

It follows that $||U^{(n)}|| = 1$ and $\lim_{n\to\infty} ||T\,U^{(n)} - \lambda\,U^{(n)}|| = 0$. Now set $u^{(n)} = \mathcal{F}\,U^{(n)}$.

[1] To prove this, suppose $A\,u - \lambda u = e^{(0)}$. Then for non-zero n, we have $u_{n-1} = \lambda\,u_n$ so $u_n = \lambda^{-1-n}\,u_{-1}$ for negative n and $u_n = \lambda^{-n}\,u_0$ for positive n. The only way for u to be in H is for u to be the zero vector which is impossible.

[2] The explicit proof is simple: For $|\lambda| > 1$ observe that $A - \lambda\,I = -\lambda\,(I - \lambda^{-1}\,A)$ and now the inverse can explicitly be constructed via a Neumann series since $||\lambda^{-1}\,A|| = |\lambda|^{-1} < 1$. Analogously, if $|\lambda| < 1$, then $A - \lambda\,I = A\,(I - \lambda\,A^*)$ which is invertible since A is invertible and since $||\lambda\,A^*|| = |\lambda| < 1$.

Problem 3: Define $T \in \mathcal{S}^*(\mathbb{R})$ via

$$\langle T, \varphi\rangle = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx.$$

Construct a continuous function f of at most polynomial growth such that $T = \partial^p f$ for some finite integer p.

Solution: First we integrate the function $1/x$ in a classical sense to find a candidate for a distributional primitive function.

$$\int\int \frac{1}{x} = \int (\log|x| + A) = x\,\log|x| - x + A\,x + B$$

Set $A = 1$ and $B = 0$ to obtain the candidate

$$f(x) = x\,\log|x|.$$

The function f is continuous and of polynomial growth. It remains to prove that $f'' = T$ in a distributional sense.

$$\begin{aligned}
\langle f'', \varphi\rangle =& \langle f, \varphi''\rangle \\
\overset{(1)}{=}& \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} f\,\varphi'' + \int_{\varepsilon}^{\infty} f\,\varphi''\right) \\
\overset{(2)}{=}& \lim_{\varepsilon\searrow 0}\left(\left[f\,\varphi'\right]_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} f'\,\varphi' + \left[f\,\varphi'\right]_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} f'\,\varphi'\right) \\
\overset{(3)}{=}& \lim_{\varepsilon\searrow 0}\left(-\int_{-\infty}^{-\varepsilon} f'\,\varphi' - \int_{\varepsilon}^{\infty} f'\,\varphi'\right) \\
\overset{(4)}{=}& \lim_{\varepsilon\searrow 0}\left(-\left[f'\,\varphi\right]_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} f''\,\varphi - \left[f'\,\varphi\right]_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} f''\,\varphi\right) \\
\overset{(5)}{=}& \lim_{\varepsilon\searrow 0}\Big(-\log(\varepsilon)\varphi(-\varepsilon) + \log(\varepsilon)\varphi(\varepsilon)\Big) + \langle T, \varphi\rangle.
\end{aligned}$$

Relation (1) holds since the integrand is continuous.
Relation (2) is plain partial integration.
Relation (3) holds since $f\,\varphi'$ is a continuous function.
Relation (4) is plain partial integration.
Relation (5) holds since $f''(x) = 1/x$ in the domains of integration.
(Note that all limits at $\pm\infty$ vanish since $f\,\varphi'$ and $f'\,\varphi$ both tend to zero since $\varphi \in \mathcal{S}$ and f and f' have at most polynomial growth.)

Finally we observe that

$$\lim_{\varepsilon\searrow 0}\big(-\log(\varepsilon)\varphi(-\varepsilon) + \log(\varepsilon)\varphi(\varepsilon)\big) = \lim_{\varepsilon\searrow 0}\log(\varepsilon)\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big) = 0$$

since

$$\big|\varphi(\varepsilon) - \varphi(-\varepsilon)\big| \leq 2\,\varepsilon\,||\varphi'||_{\mathrm{u}}$$

and

$$\lim_{\varepsilon\searrow 0}\varepsilon\,\log(\varepsilon) = 0.$$

Problem 4: Fix $\psi \in \mathcal{S}(\mathbb{R})$. Define the map

$$B:\ \mathcal{S}(\mathbb{R}) \to \mathbb{C}:\ \varphi \mapsto \int_{\mathbb{R}} \psi(x)\,\varphi'(x)\,dx.$$

Prove that B is continuous. What order is B?

Solution:

First observe that via a partial integration we can rewrite

$$B(\varphi) = -\int_{-\infty}^{\infty} \psi'(x)\,\varphi(x)\,dx.$$

Then

$$|B(\varphi)| = \left| -\int_{-\infty}^{\infty} \psi'(x)\,\varphi(x)\,dx \right| \le \int_{-\infty}^{\infty} |\psi'(x)|\,|\varphi(x)|\,dx \le ||\psi'||_{L^1}\,||\varphi||_{0,0}.$$

Observe that $||\psi'||_{L^1}$ is finite[3] since $\psi \in \mathcal{S}$ so B is continuous and has order zero.

[3] To be precise $||\psi'||_{L^1} = \int |\psi'| \le \int (1+x^2)\,||\psi||_{0,2} = \pi\,||\psi||_{0,2} < \infty$.

Problem 5: Set $H = L^2(\mathbb{T})$ and define $W \in \mathcal{B}(H)$ via

$$[W\,u](x) = \int_{-\pi}^{\pi} \sin(x-y)\,u(y)\,dy.$$

Compute the spectrum of W and identify its different components (i.e. determine $\sigma_{\rm p}(W)$, $\sigma_{\rm c}(W)$, and $\sigma_{\rm r}(W)$). Is W compact? Self-adjoint?

Solution: We define the canonical basis for H via

$$e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \qquad n \in \mathbb{Z},$$

and the corresponding canonical projections P_n via

$$[P_n u](x) = e_n(x)\,\langle e_n,\,u\rangle = \frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} e^{-iny}u(y)\,dy.$$

Then observe that W can be written

$$\begin{aligned}[W\,u](x) &= \int_{-\pi}^{\pi} \frac{e^{i(x-y)} - e^{-i(x-y)}}{2i}\,u(y)\,dy \\ &= \frac{e^{ix}}{2i}\int_{-\pi}^{\pi} e^{-iy}\,u(y)\,dy - \frac{e^{-ix}}{2i}\int_{-\pi}^{\pi} e^{iy}\,u(y)\,dy = -i\pi[P_1 u](x) + i\pi[P_{-1}u](x).\end{aligned}$$

It follows that

$$\sigma(W) = \sigma_p(W) = \{0,\,i\pi,\,-i\pi\},$$

and consequently $\sigma_{\rm c}(W) = \sigma_{\rm r}(W) = \emptyset$.

Alternative solution: Recalling the trig identity

$$\sin(x-y) = \sin(x)\,\cos(y) - \cos(x)\,\sin(y)$$

we write

$$[W\,u](x) = \sin(x)\int_{-\pi}^{\pi}\cos(y)\,u(y)\,dy - \cos(x)\int_{-\pi}^{\pi}\sin(y)\,u(y)\,dy.$$

Defining two orthonormal unit vectors v_1 and v_2 via

$$v_1(x) = \frac{1}{\sqrt{\pi}}\sin(x), \qquad v_2(x) = \frac{1}{\sqrt{\pi}}\cos(x),$$

we can therefore write W as

$$W\,u = \pi\,v_1\,\langle v_2,\,u\rangle - \pi\,v_2\,\langle v_1,\,u\rangle.$$

Now set $G = \text{span}\{v_1,\,v_2\}$ and observe that both G and $G^{\perp}$ are invariant subspaces of W. The restriction of W to G has the matrix

$$\mathrm{W} = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$$

and W has the eigenvalues $\pm i\pi$. The restriction of W to $G^{\perp}$ is zero. Therefore

$$\sigma(W) = \sigma_p(W) = \{0,\,i\pi,\,-i\pi\}.$$