Applied Analysis (APPM 5450): Midterm 2

8.30am – 9.50am, Mar. 14, 2011. Closed books.

The problems are worth 20 points each. Briefly motivate all answers except those to Problem 1.

Problem 1: No motivation required for these questions.

(a) Give an example of a bounded linear operator on a Hilbert space that is positive, but not coercive.

(b) Let H be an infinite dimensional Hilbert space. Which of the following sets can be the spectrum of a compact self-adjoint operator?
(1) $A_1 = \{1/n\}_{n=1}^{\infty} = \{1,\, 1/2,\, 1/3,\, 1/4,\, \dots\}$
(2) $A_2 = \{1\} \cup \{1 - 1/n\}_{n=1}^{\infty} = \{1,\, 0,\, 1/2,\, 2/3,\, 3/4,\, 4/5,\, \dots\}$.
(3) $A_3 = \{0,\, 1\} \cup \{e^{i/n}\}_{n=1}^{\infty} = \{0,\, 1,\, e^{i},\, e^{i/2},\, e^{i/3},\, e^{i/4},\, \dots\}$.
(4) $A_4 = \{1,\, 2,\, 3\}$.
(5) $A_5 = \{-1,\, 0\}$.

(c) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}$. What is $\langle \delta'',\, \varphi \rangle$?

(d) Define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = e^{-x^2}$. What is $\delta'' * \varphi$?

Problem 2: Set $H = \ell^2(\mathbb{Z})$ and let $A \in \mathcal{B}(H)$ denote the rightshift operator (i.e. if $u \in H$ and $v = A\,u$, then $v_n = u_{n-1}$).

(a) Let λ be a complex number such that $|\lambda| = 1$. Prove that you can construct $u^{(n)} \in H$ such that $||u^{(n)}|| = 1$ and $\lim_{n\to\infty} ||A\,u^{(n)} - \lambda\,u^{(n)}|| = 0$.

(b) Determine the spectrum of A.

Problem 3: Define $T \in \mathcal{S}^*(\mathbb{R})$ via

$$\langle T,\, \varphi \rangle = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx.$$

Construct a continuous function f of at most polynomial growth such that $T = \partial^p f$ for some finite integer p.

Problem 4: Fix $\psi \in \mathcal{S}(\mathbb{R})$. Define the map

$$B :\ \mathcal{S}(\mathbb{R}) \to \mathbb{C} :\ \varphi \mapsto \int_{\mathbb{R}} \psi(x)\,\varphi'(x)\,dx.$$

Prove that B is continuous. What order is B?

Problem 5: Set $H = L^2(\mathbb{T})$ and define $W \in \mathcal{B}(H)$ via

$$[W u](x) = \int_{-\pi}^{\pi} \sin(x - y)\,u(y)\,dy.$$

Compute the spectrum of W and identify its different components (i.e. determine $\sigma_{\rm p}(W)$, $\sigma_{\rm c}(W)$, and $\sigma_{\rm r}(W)$). Is W compact? Self-adjoint?