Applied Analysis (APPM 5450): Midterm 1 — Solutions

8.30am – 9.50am, Feb. 14, 2011. Closed books.

Problem 1: (21p) All operators in this problem are bounded linear operators on a Hilbert space. Which statements are necessarily true? No motivation required.

(a) Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

(b) If A and B are self-adjoint operators, then $A+B$ is self-adjoint.

(c) If A and B are self-adjoint operators, then $A\,B$ is self-adjoint.

(d) If A and B are unitary operators, then $A+B$ is unitary.

(e) If A and B are unitary operators, then $A\,B$ is unitary.

(f) If A is skew-symmetric, then the operator $B = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is unitary.

(g) If A is an isometric operator, then $\text{ran}(A) = \big(\ker(A^*)\big)^{\perp}$.

Solution:

(a) TRUE. (Observe that the unit ball in a Hilbert space is weakly compact.)

(b) TRUE. (A simple calculation will demonstrate this.)

(c) FALSE. (Counter example: $H = \mathbb{C}^2$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.)

(d) FALSE. (Counter example: $A = I$, $B = I$, $A + B = 2I$.)

(e) TRUE. (Note that for any $x, y \in H$, we have $\langle ABx,\, ABy\rangle = \langle Bx,\, By\rangle = \langle x,\, y\rangle$ where we first used that A is unitary, and then used that B is unitary. So AB preserves the inner product (and is therefore 1-to-1). To prove that it is onto, consider $z \in H$. Since B is onto, there is a $y \in H$ such that $z = By$. Since A is onto, there is an $x \in H$ such that $y = Ax$. Therefore, $z = ABx$.)

(f) TRUE. (You can prove that $B^* = B^{-1}$ by simply taking the adjoint of the definition.)

(g) TRUE. (Note that when A is an isometry, $\text{ran}(A)$ is necessarily closed.)

Problem 2: (29p) Let H_1 denote the Hilbert space obtained by taking the completion of the set $\mathcal{P}$ of trigonometric polynomials with respect to the norm induced by the inner product

$$\langle u,\, v\rangle_1 = \int_{-\pi}^{\pi} \overline{u(x)}\, v(x)\, dx$$

and let H_2 denote the Hilbert space induced by the inner product

$$\langle u,\, v\rangle_2 = \int_{-\pi}^{\pi} \overline{u(x)}\, v(x)\, (1-\cos(x))\, dx.$$

(a) Do the spaces H_1 and H_2 contain the same [equivalence classes of] functions?
(b) Does there exist a unitary map between H_1 and H_2?
(c) For which real numbers α is it the case that the sequence $(\varphi_n)_{n=1}^{\infty}$ where $\varphi_n = n^{\alpha}\,\chi_{(-1/n,\,1/n)}$ converges in norm in H_1? Is the answer different if you consider weak convergence?
(d) Repeat question (c), but now do the exercise in H_2.
(e) Set $\rho_n(x) = \sin(nx)$. Does $(\rho_n)_{n=1}^{\infty}$ converge in either H_1 or H_2? Weakly? In norm?

Solution: First observe that $1-\cos x = 2\big(\sin(x/2)\big)^2 \sim (1/2)x^2$ as $x \to 0$. Also note that for both $i=1$ and $i=2$ we know that $\mathcal{P}$ is dense in H_i so a sequence $w_n \rightharpoonup w$ in H_i iff (a) $\sup_n ||w_n||_i < \infty$ and (b) $\langle w_n, u\rangle_i \to \langle w, u\rangle_i$ for all $u \in \mathcal{P}$.

(a). No. Set $u(x) = 1/x$. Then $||u||_1^2 = \int_{-\pi}^{\pi} x^{-2}\, dx = \infty$ but $||u||_2^2 = \int_{-\pi}^{\pi} x^{-2} 2\,(\sin(x/2))^2\, dx \le \int_{-\pi}^{\pi}(1/2)\, dx = \pi$.

(b) Yes. H_1 and H_2 are both separable Hilbert spaces so they are unitarily equivalent. (You can explicitly construct orthonormal bases for the two spaces by performing Gram-Schmidt with respect to the two inner products to the sequence $\{1,\, \cos(x),\, \sin x,\, \cos(2x),\, \sin(2x),\, \cos(3x),\, \dots\}$.)

Alternatively, you can prove that the map $U :\, H_2 \to H_1$ with $[U\, f](x) = \sqrt{1-\cos(x)}\, f(x)$ is an isometric bijection.

(c) $||\varphi_n||_1^2 = \int_{-1/n}^{1/n} n^{2\alpha}\, dx = 2\, n^{2\alpha-1}$. If $\alpha > 1/2$ then the sequence is unbounded and can converge neither weakly nor in norm. If $\alpha < 1/2$ then $\varphi_n \to 0$ in norm (and hence weakly as well). For the case $\alpha = 1/2$, observe (a) that the sequence is bounded, and (b) that for $u \in \mathcal{P}$ we have $|\langle \varphi_n,\, u\rangle_1| = |\int_{-1/n}^{1/n} n^{1/2} u(x)\, dx| \le \big(\sup_x |u(x)|\big) \int_{-1/n}^{1/n} n^{1/2}\, dx = \big(\sup_x |u(x)|\big)\, 2\, n^{-1/2} \to 0$ so $\varphi_n \rightharpoonup 0$.

(d) $||\varphi_n||_2^2 = 2\int_0^{1/n} n^{2\alpha}(1-\cos x)\, dx = 2\, n^{2\alpha}\,\big((1/n) - \sin(1/n)\big) \sim n^{2\alpha-3}$. If $\alpha > 3/2$ then (φ_n) is unbounded and can converge neither weakly nor in norm. If $\alpha < 3/2$ then $\varphi_n \to 0$ in norm (and hence weakly as well). When $\alpha = 3/2$, observe (a) that the sequence is bounded, and (b) that for $u \in \mathcal{P}$ we have $|\langle \varphi_n,\, u\rangle_2| \le \int_{-1/n}^{1/n} n^{3/2}|u(x)|(1/2)x^2\, dx \le \big(\sup_x |u(x)|\big) \int_0^{1/n} n^{3/2}\, x^2\, dx = \big(\sup_x |u(x)|\big)\, (1/3)\, n^{-3/2} \to 0$ so $\varphi_n \rightharpoonup 0$.

(e) First observe that (ρ_n) is bounded in both spaces. Next fix a function $v \in \mathcal{P}$. We have

$$(1) \qquad \left|\int_{-\pi}^{\pi} \sin(nx)\, v(x)\, dx\right| = \left|\frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)\, v'(x)\, dx\right| \le \frac{1}{n}||v'||_{L^1(\mathbb{T})} \to 0 \quad \text{as } n \to 0$$

since the boundary conditions in the partial integration vanish due to periodicity. This immediately shows that $\langle \rho_n, v\rangle_1 \to 0$ for all $v \in \mathcal{P}$ and therefore $\rho_n \rightharpoonup 0$ in H_1. We also have $\langle \rho_n, u\rangle_2 \to 0$ for all $u \in \mathcal{P}$ since (1) holds for $v(x) = u(x)\,(1-\cos x)$. Therefore $\rho_n \rightharpoonup 0$ in H_2. To see that (ρ_n) cannot converge in norm to zero, simply note that $||\rho_n||_1^2 = \pi$ and

$$||\rho_n||_2^2 = \int_{-\pi}^{\pi} \sin^2(nx)(1-\cos(x))\, dx = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}(1-\cos(x))\, dx = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\, dx = \pi.$$

Problem 3: (20p) Set $f(t) = |t|$ for $-\pi \leq t < \pi$ and extend f to be a 2π-periodic function. Is it the case that $f \in H^k(\mathbb{T})$ for any $k \geq 0$?

Hint: The Sobolev embedding theorem should very quickly provide at least a partial answer.

Solution: First we construct the Fourier expansion of f. Since f is real and even, its expansion consists of cosines only:

$$f(x) = \beta_0 + \sum_{n=1}^{\infty} \beta_n \cos(nx).$$

The constant is the average of f so $\beta_0 = \pi/2$. To determine β_n multiply both sides by $\cos(nx)$ and integrate:

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \beta_n \int_{-\pi}^{\pi} \big(\cos(nx)\big)^2\,dx.$$

We have

$$\int_{-\pi}^{\pi} \big(\cos(nx)\big)^2\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}\,dx = \pi,$$

and

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 2\int_0^{\pi} x\cos(nx)\,dx = 2\left[\frac{1}{n}x\,\sin(nx)\right]_0^{\pi} - 2\int_0^{\pi}\frac{1}{n}\sin(nx)\,dx$$
$$= 2\left[\frac{1}{n^2}\cos(nx)\right]_0^{\pi} = \frac{2}{n^2}\big(\cos(n\pi)-1\big) = \frac{2}{n^2}\big((-1)^n - 1\big).$$

So

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\dots}\frac{1}{n^2}\cos(nx) = \sum_{n\in\mathbb{Z}} \alpha_n \frac{e^{inx}}{\sqrt{2\pi}}$$

where

$$\alpha_n = \begin{cases} \pi^{3/2}\,2^{-1/2} & n = 0 \\ 0 & n = \pm 2, \pm 4, \pm 6, \dots \\ 2^{5/2}\,\pi^{-1/2}\,n^{-2} & n = \pm 1, \pm 3, \pm 5, \dots \end{cases}$$

We find that

$$||f||_{H^k}^2 = \frac{\pi^3}{2} + \sum_{n\ \text{odd}} |n|^{2k}\frac{32}{\pi\,n^4}$$

The sum is convergent if and only if $2k - 4 < -1$, which is to say if $k < 3/2$.

Answer: $f \in H^k$ if and only if $k < 3/2$.

Partial answer alluded to in hint: Observe that $f \notin C^1$. If $k > 3/2$, then $H^k \subseteq C^1$, and since $f \notin C^1$, it follows that $f \notin H^k$ when $k > 3/2$.

Problem 4: (30p) Suppose that P is a projection on a Hilbert space H. Prove that the following are equivalent:

(i) P is orthogonal, *i.e.* $\ker(P) = \text{ran}(P)^{\perp}$.

(ii) P is self-adjoint, *i.e.* $\langle P\,x,\, y\rangle = \langle x,\, P\,y\rangle \quad \forall x, y$.

(iii) $||P|| = 0$ or 1.

Solution:

(a) $\Rightarrow$ (b): Assume $\ker(P) = \text{ran}(P)^{\perp}$. Pick any $x, y \in H$. Then

$$(Px,\, y) = (\underbrace{Px}_{\in \text{ran}(P)},\, Py + \underbrace{(I-P)y}_{\in \ker(P)}) = (Px,\, Py) = (Px + (I-P)x,\, Py) = (x,\, Py).$$

(b) $\Rightarrow$ (c): Assume that (b) holds. Then for any x,

$$||Px||^2 = (Px,\, Px) = (P^2x,\, x) = (Px,\, x) \leq ||Px||\,||x||,$$

so $||P|| \leq 1$. Obviously it is possible for $||P||$ to be zero. We need to prove that the only possible non-zero value of $||P||$ is one. To this end, note that if $P \neq 0$, then $\text{ran}(P) \neq \{0\}$. Now observe that if x is a non-zero element in $\text{ran}(P)$, we have $Px = x$ so $||P|| \geq 1$.

(c) $\Rightarrow$ (a): Assume that (a) does not hold. Then there exist $x \in \text{ran}(P)$ and $y \in \ker(P)$ such that $(x,\, y) \neq 0$. Set $\alpha = \overline{(x,\, y)}/|(x,\, y)|$ and $z = \alpha y$. Then $z \in \ker(P)$ and $(x,\, z) = |(x,\, y)| \in \mathbb{R}_{+}$. Set

$$w = x - z\,t.$$

Then $||Pw|| = ||x||$, and

$$||w||^2 = ||x||^2 - 2\,t\,(x,\, z) + t^2\,||z||^2.$$

For small t, we see that $||w|| < ||x|| = ||Pw||$ so $||P|| > 1$.