Applied Analysis (APPM 5450): Midterm 1

8.30am – 9.50am, Feb. 14, 2011. Closed books.

Problem 1: (21p) All operators in this problem are bounded linear operators on a Hilbert space. Which statements are necessarily true? No motivation required.

(a) Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

(b) If A and B are self-adjoint operators, then $A + B$ is self-adjoint.

(c) If A and B are self-adjoint operators, then $A\,B$ is self-adjoint.

(d) If A and B are unitary operators, then $A + B$ is unitary.

(e) If A and B are unitary operators, then $A\,B$ is unitary.

(f) If A is skew-symmetric, then the operator $B = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is unitary.

(g) If A is an isometric operator, then $\mathrm{ran}(A) = \big(\ker(A^*)\big)^{\perp}$.

Problem 2: (29p) Let H_1 denote the Hilbert space obtained by taking the completion of the set $\mathcal{P}$ of trigonometric polynomials with respect to the norm induced by the inner product

$$\langle u,\, v\rangle_1 = \int_{-\pi}^{\pi} \overline{u(x)}\, v(x)\, dx$$

and let H_2 denote the Hilbert space induced by the inner product

$$\langle u,\, v\rangle_2 = \int_{-\pi}^{\pi} \overline{u(x)}\, v(x)\, (1 - \cos(x))\, dx.$$

(a) Do the spaces H_1 and H_2 contain the same [equivalence classes of] functions?

(b) Does there exist a unitary map between H_1 and H_2?

(c) For which real numbers α is it the case that the sequence $(\varphi_n)_{n=1}^{\infty}$ where $\varphi_n = n^{\alpha}\,\chi_{(-1/n,\,1/n)}$ converges in norm in H_1? Is the answer different if you consider weak convergence?

(d) Repeat question (c), but now do the exercise in H_2.

(e) Set $\rho_n(x) = \sin(nx)$. Does the sequence $(\rho_n)_{n=1}^{\infty}$ converge in either H_1 or H_2? Weakly? In norm?

Problem 3: (20p) Set $f(t) = |t|$ for $-\pi \leq t < \pi$ and extend f to be a 2π-periodic function. Is it the case that $f \in H^k(\mathbb{T})$ for any $k \geq 0$?

Hint: The Sobolev embedding theorem should very quickly provide at least a partial answer.

Problem 4: (30p) Suppose that P is a projection on a Hilbert space H. Prove that the following are equivalent:

(i) P is orthogonal, *i.e.* $\ker(P) = \mathrm{ran}(P)^{\perp}$.

(ii) P is self-adjoint, *i.e.* $\langle P\,x,\, y\rangle = \langle x,\, P\,y\rangle \quad \forall x, y$.

(iii) $||P|| = 0$ or 1.