Homework 9

11.4) If $\phi \in S(R)$, prove that $\phi\delta' = \phi(0)\delta' - \phi'(0)\delta$.

$$\langle \phi\delta', \psi \rangle = \langle \delta', \phi\psi \rangle = -\left\langle \delta, (\phi\psi)' \right\rangle = -\langle \delta, \phi'\psi + \phi\psi' \rangle = -\langle \delta, \phi'\psi \rangle - \langle \delta, \phi\psi' \rangle = -\phi'(0)\psi(0) - \phi(0)\psi'(0) =$$
$$= \langle -\phi'(0)\delta, \psi \rangle + \langle \phi(0)\delta', \psi \rangle = \langle -\phi'(0)\delta + \phi(0)\delta', \psi \rangle$$

Note that in the equality across the line break the following was used: $-\psi'(0) = -\langle \delta, \psi' \rangle = \langle \delta', \psi \rangle$

11.9) Let $\psi \in S$ and define the convolution operator $Kf(x) = \int \psi(x-y)f(y)dy$ for all $f \in S$. Prove that $K : S \to S$ is a continuous linear operator for the topology of S.

Pick $\phi \in S$. Then

$$\|K\phi\|_{\alpha,k} = \sup_x \left| \left(1+|x|^2\right)^{k/2} \partial_x^\alpha \int \psi(x-y)\phi(y)dy \right| = \sup_x \left| \left(1+|x|^2\right)^{k/2} \int \psi^{(\alpha)}(x-y)\phi(y)dy \right| = (1)$$

Next we introduce the substitution $z = y - \frac{x}{2}$

$$(1) = \sup_x \left| \left(1+|x|^2\right)^{k/2} \int \psi^{(\alpha)}\left(\frac{x}{2}-z\right)\phi\left(\frac{x}{2}+z\right)dy \right| \le \sup_x \left| \left(1+|x|^2\right)^{k/2} \int \frac{\|\psi\|_{\alpha,2N}}{\left(1+\left|\frac{x}{2}-z\right|^2\right)^{\frac{N}{2}}} \frac{\|\phi\|_{0,2N}}{\left(1+\left|\frac{x}{2}+z\right|^2\right)^{\frac{N}{2}}} dz \right| = (2)$$

Note that in the step above where the $\le$ is we pick N s.t. $N \ge k, N \ge d+1$.
We can bound the denominator (not including the exponent) as follows

$$\left(1+\left|\frac{x}{2}-z\right|^2\right)\left(1+\left|\frac{x}{2}+z\right|^2\right) = 1 + \frac{1}{16}|x|^4 + |z|^4 + \frac{1}{2}|x|^2 + 2|z|^2 + \frac{1}{2}|x|^2|z|^2 - \underbrace{|x \cdot z|}_{\le |x||z|}{}^2 \ge$$

$$\ge 1 + \frac{1}{2}|x|^2 + 2|z|^2 + \underbrace{\frac{1}{16}|x|^4 + |z|^4 - \frac{1}{2}|x|^2|z|^2}_{=\left(\frac{1}{4}|x|^2 - |z|^2\right)^2 \ge 0} \ge 1 + \frac{1}{2}|x|^2 + 2|z|^2$$

Continuing from above we have

$$(2) \le \sup_x \left| \left(1+|x|^2\right)^{k/2} \int \frac{\|\psi\|_{\alpha,2N}}{\left(1+\frac{1}{2}|x|^2+2|z|^2\right)^{\frac{N}{2}}} \frac{\|\phi\|_{0,2N}}{\left(1+\frac{1}{2}|x|^2+2|z|^2\right)^{\frac{N}{2}}} dz \right| \le$$

$$\le \sup_x \left| \left(1+|x|^2\right)^{k/2} \int \frac{\|\psi\|_{\alpha,2N}}{\left(1+\frac{1}{2}|x|^2\right)^{\frac{N}{2}}} \frac{\|\phi\|_{0,2N}}{\left(1+2|z|^2\right)^{\frac{N}{2}}} dz \right| = C\|\psi\|_{\alpha,2N}\|\phi\|_{0,2N}$$

Combining everything we have $\|K\phi\|_{\alpha,k} \le C\|\psi\|_{\alpha,2N}\|\phi\|_{0,2N}$

Thus $K : S \to S$ is a continuous linear operator for the topology of S.

11.10) For every $h \in R^n$ define a linear transform $\tau_h : S \to S$ by $\tau_h(f)(x) = f(x-h)$.

a) Prove that for all $h \in R^n$, τ_h is continuous in the topology of S.

Assume $\phi_n \to \phi$ in S.

$$\|\tau_h(\phi_n) - \tau_h(\phi)\|_{\alpha,k} = \sup_x \left|\left(1+|x|^2\right)^{k/2}\left(\partial^\alpha \phi_n(x-h) - \partial^\alpha \phi(x-h)\right)\right| = \sup_x \left|\left(1+|x+h|^2\right)^{k/2}\left(\partial^\alpha \phi_n(x) - \partial^\alpha \phi(x)\right)\right| = (*)$$

Where the final equality above substitutes $x+h$ for x.

We can bound this as follows

$$1+|x+h|^2 \le 1+\left(|x|+|h|\right)^2 \le 1+|x|^2+|h|^2+2\underbrace{|x||h|}_{\le |x|^2+|h|^2} \le 1+2|x|^2+2|h|^2 \le 2\left(1+|x|^2\right)\left(1+|h|^2\right)$$

Using this bound and continuing from (*) we have

$$\|\tau_h(\phi_n) - \tau_h(\phi)\|_{\alpha,k} = \cdots = (*) \le \sup_x \left|\left(2\left(1+|x|^2\right)\left(1+|h|^2\right)\right)^{k/2}\left(\partial^\alpha \phi_n(x) - \partial^\alpha \phi(x)\right)\right| = \left(2\left(1+|h|^2\right)\right)^{k/2}\|\phi_n - \phi\|_{\alpha,k} \to 0$$

The convergence is in the last step comes from the assumption that $\phi_n \to \phi$ in S.

b) Prove that for all $f \in S$, the map $h \mapsto \tau_h f$ is continuous from R^n to S.

Assume $h \to 0$ in R^d. Then

$$\|\tau_h\phi - \phi\|_{\alpha,k} = \left\|\left(1+|x|^2\right)^{k/2}\partial^\alpha\left(\phi(x-h) - \phi(x)\right)\right\| \le |h|\left\|\left(1+|x|^2\right)^{k/2}\nabla\partial^\alpha\phi(x_n)\right\|_u \le |h|C\sum_{|\beta|=|\alpha|+1}\|\phi\|_{\beta,k} \xrightarrow{h\to 0} 0$$

Note that above x_n is some point on the line from $x-h$ to x and the first inequality uses the mean value theorem for integrals.

Problem 1) We say that a sequence $(\phi_n)_{n=1}^{\infty}$ is an approximate identity if

1) $\phi_n \in C(R^d),\ \forall n$
2) $\phi_n(x) \geq 0,\ \forall n, x$
3) $\int_{R^d} \phi_n(x)dx = 1,\ \forall n$
4) $\forall \varepsilon > 0,\ \int_{|x|\geq\varepsilon} \phi_n(x)dx \xrightarrow{n\to\infty} 0$

a) Do the conditions imply that $\phi_n \in S^*$?

Yes. Conditions (1)-(3) above imply that $\phi_n \in L^1$, and this immediately implies $\phi_n \in S^*$.

b) Assuming that $\phi_n \in S^*$, prove that $\phi_n \to \delta$ in S^*. Fix $\varepsilon > 0$.

$$\left|\langle \phi_n, \phi\rangle - \phi(0)\right| = \left|\int_{R^d} \phi_n(x)\phi(x)dx - \phi(0)\right| \overset{(a)}{=} \left|\int_{R^d} \phi_n(x)(\phi(x) - \phi(0))dx\right| =$$

$$= \left|\int_{|x|<\varepsilon} \phi_n(x)(\phi(x)-\phi(0))dx + \int_{|x|\geq\varepsilon} \phi_n(x)(\phi(x)-\phi(0))dx\right| \leq \int_{|x|<\varepsilon} \phi_n(x)\underbrace{|\phi(x)-\phi(0)|}_{\leq\varepsilon\|\nabla\phi\|_u = \varepsilon\|\phi\|_{1,0}}dx + \int_{|x|\geq\varepsilon} \phi_n(x)\underbrace{|\phi(x)-\phi(0)|}_{\leq|\phi(x)|+|\phi(0)|\leq 2\|\phi\|_u}dx$$

Note that the equality denoted by (a) above uses condition (3).

We now have $\left|\langle \phi_n, \phi\rangle - \phi(0)\right| \leq \varepsilon\|\phi\|_{1,0} + 2\|\phi\|_u \underbrace{\int_{|x|\geq\varepsilon} \phi_n(x)dx}_{\xrightarrow{n\to\infty} 0}$

This implies $\limsup_{n\to\infty} \left|\langle \phi_n, \phi\rangle - \phi(0)\right| \leq \varepsilon\|\phi\|_{1,0}$

Since ε was arbitrary we get $\left|\langle \phi_n, \phi\rangle - \phi(0)\right| \to 0$, or simply $\phi_n \to \delta$ in S^*.

Problem 3) Let k be a positive integer. Prove that there exist c_k, C_k s.t. $0 < c_k \le C_k < \infty$, and

(1) $$c_k\left(1+|x|^k\right) \le \left(1+|x|^2\right)^{k/2} \le C_k\left(1+|x|^k\right), \qquad \forall x \in R^d$$

Prove that there exist b_k, B_k s.t. $0 < b_k \le B_k < \infty$, and

(2) $$b_k\left(1+|x|\right)^k \le \left(1+|x|^2\right)^{k/2} \le B_k\left(1+|x|\right)^k, \qquad \forall x \in R^d$$

To prove (1) we need to prove the following

$$(a) \quad \sup_{x\in R^d} \frac{\left(1+|x|^2\right)^{k/2}}{\left(1+|x|^k\right)} < \infty \quad \Leftrightarrow \quad \sup_{0\le r<\infty} \frac{\left(1+r^2\right)^{k/2}}{\left(1+r^k\right)} < \infty$$

$$(b) \quad \inf_{x\in R^d} \frac{\left(1+|x|^2\right)^{k/2}}{\left(1+|x|^k\right)} > 0 \quad \Leftrightarrow \quad \inf_{0\le r<\infty} \frac{\left(1+r^2\right)^{k/2}}{\left(1+r^k\right)} > 0$$

Set $f(r) = \frac{\left(1+r^2\right)^{k/2}}{\left(1+r^k\right)}$. Then $f(0)=1$ and $f(\infty)=1$.

Since f is continuous and $f(0)=1$ and $f(\infty)=1$, the supremum and infemum of f are attained. Since $0 < f(r) < \infty$, it follows that $\sup_{0\le r<\infty} f(r) < \infty$ and $\inf f(r) > 0$.

The proof for (2) is similar.