Homework 7

9.19) Let A be a compact, self-adjoint operator with spectral decomposition $A = \sum_{n=1}^{\infty} \lambda_n P_n$.

Prove that $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$ is convergent in the strong operator topology for any $f \in C(\sigma(A))$, and that it converges uniformly if we also have $f(0) = 0$.

Fix an x. For strong convergence we need to prove that $\left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n x \right\| \xrightarrow{N\to\infty} 0$.

We have $\lim_{N\to\infty} \left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n x \right\| \le \lim_{N\to\infty} \left(\underbrace{\sup_{n\ge N+1} |f(\lambda_n)|}_{=M<\infty} \cdot \left\| \sum_{n=N+1}^{\infty} P_n x \right\| \right)$.

Now we use that the P_n's are orthogonal so $\left\| \sum_{n=N+1}^{\infty} P_n x \right\| = \left(\sum_{n=N+1}^{\infty} \|P_n x\|^2 \right)^{1/2}$ which converges to zero as N goes to infinity since $\sum_{n=1}^{\infty} \|P_n x\|^2 \le \|x\|^2$.

Next assume that $f(0) = 0$. For uniform convergence we need $\left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n \right\| \xrightarrow{N\to\infty} 0$.

Since A is compact we have $\lambda_n \to 0$ and since f is continuous we have $f(\lambda_n) \to f(0) = 0$.

Then $\lim_{N\to\infty} \left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n \right\| \le \lim_{N\to\infty} \left(\sup_{n\ge N+1} |f(\lambda_n)| \cdot \underbrace{\left\| \sum_{n=N+1}^{\infty} P_n \right\|}_{=1} \right) = \lim_{N\to\infty} \sup_{n\ge N+1} |f(\lambda_n)| = 0$.

9.20) Let A be a self-adjoint, compact operator on a Hilbert space H and let $f:\sigma(A)\to C$ be a continuous function. When is $f(A)$ compact?

Let A have the spectral decomposition $A=\sum_{n=1}^{\infty}\lambda_n P_n$.

Then define $B=f(A)=\sum_{n=1}^{\infty}f(\lambda_n)P_n$. We then show that B is a compact operator if and only if $f(0)=0$.

$\Leftarrow$) Assume $f(0)=0$. Set $B_N=\sum_{n=1}^{N}f(\lambda_n)P_n$. Then B_N is of finite rank and $\|B_N-B\|\xrightarrow{N\to\infty}0$ so B is a compact operator.

$\Rightarrow$) Assume B is a compact operator. Define ϕ_n s.t. $A\phi_n=\lambda_n\phi_n$ and $\|\phi_n\|=1$. Since B is a compact operator $(B\phi_n)$ has a convergent subsequence, but if $f(0)\neq 0$ then this is impossible because $(B\phi_n)$ is an orthogonal sequence and for sufficiently large n we have $\|B\phi_n\|\geq\frac{|f(0)|}{2}>0$.

Note: Above we used the fact that if an orthogonal sequence converges then it must converge to zero.

9.22) Suppose that A is a compact, nonnegative linear operator on a Hilbert space H. Prove that there is a unique compact, nonnegative linear operator B s.t. $B^2=A$. Thus, $B=A^{1/2}$ is the square root of A.

Note that an operator A is nonnegative if it is self-adjoint and $\langle x,Ax\rangle\geq 0$ for all $x\in H$.

Let A have the spectral decomposition $A=\sum_{n=1}^{\infty}\lambda_n P_n$.

Then all $\lambda_n's$ are nonnegative (since if $\lambda_n<0$ then $\langle A\phi_n,\phi_n\rangle=\lambda_n\|\phi_n\|^2<0$).

Set $B=\sum_{n=1}^{\infty}\sqrt{\lambda_n}P_n$.

Then $BB=\sum_{n=1}^{\infty}\sqrt{\lambda_n}P_n\sum_{m=1}^{\infty}\sqrt{\lambda_m}P_m=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sqrt{\lambda_n}\sqrt{\lambda_m}\underbrace{P_nP_m}_{=\delta_{n,m}P_n}=\sum_{n=1}^{\infty}\lambda_nP_n=A$

11.1) Let X be a locally convex space. Prove the following:
b) A topology defined by a family of seminorms has a base of convex open neighborhoods. Such a topological space is called locally convex.

Define $B_{x,\alpha,\varepsilon} = \{y : p_\alpha(x-y) < \varepsilon\}$. Then a basis is given by the sets $\bigcap_{j=1}^{n} B_{x_j,\alpha_j,\varepsilon_j}$.

First we show that all $B_{x,\alpha,\varepsilon}$ are convex: Assume $y_0, y_1 \in B_{x,\alpha,\varepsilon}$. We need to show that $y_\delta = \delta y_1 + (1-\delta)y_0 \in B_{x,\alpha,\varepsilon}$ for all $\delta \in [0,1]$.

$$p_\alpha(y_\delta - x) = p_\alpha(\delta y_1 + (1-\delta)y_0 - x) = p_\alpha(\delta(y_1 - x) + (1-\delta)(y_0 - x)) \le \delta \underbrace{p_\alpha(y_1 - x)}_{<\varepsilon} + (1-\delta)\underbrace{p_\alpha(y_0 - x)}_{<\varepsilon} <$$

$$< \delta\varepsilon + (1-\delta)\varepsilon = \varepsilon$$

The first equality substitutes in the definition of y_δ.
The second equality regroups.
The $\le$ uses the triangle identity combined with the linearity of the seminorms.
The strict inequality (across the page) uses the fact that $y_0, y_1 \in B_{x,\alpha,\varepsilon}$ so each of the notes terms is less than ε.
The final equality is self-explanatory.
We have now shown that all $B_{x,\alpha,\varepsilon}$ are convex.

Now if $y_0, y_1 \in \bigcap_{j=1}^{n} B_{x_j,\alpha_j,\varepsilon_j}$ then $y_0, y_1 \in B_{x_j,\alpha_j,\varepsilon_j}$ for all j (because it is a set of intersections).
This implies that $\delta y_1 + (1-\delta)y_0 \in B_{x_j,\alpha_j,\varepsilon_j}$ for all j (proved above).
This in turn implies that $\delta y_1 + (1-\delta)y_0 \in \bigcap_{j=1}^{n} B_{x_j,\alpha_j,\varepsilon_j}$.

c) If for all $x \in X$ there exists $\alpha \in A$ s.t. $p_\alpha(x) > 0$, then the topology defined by $\{p_\alpha \mid \alpha \in A\}$ is Hausdorff.

Assume that for all x there exists an α s.t. $p_\alpha(x) \ne 0$. We need to prove that for all $x, y \in X$ there exist open sets G, H s.t. $G \cap H \ne \{empty\}$ and $x \in G, y \in H$.
By assumption there exists an α s.t. $p_\alpha(x-y) = \varepsilon > 0$. Set $G = B_{x,\alpha,\varepsilon/3}, H = B_{y,\alpha,\varepsilon/3}$.
If $z \in G$ then

$$p_\alpha(x-y) \le p_\alpha(x-z) + p_\alpha(z-y) \Rightarrow p_\alpha(z-y) \ge \underbrace{p_\alpha(x-y)}_{=\varepsilon} - \underbrace{p_\alpha(x-z)}_{<\varepsilon/3} \ge \varepsilon - \varepsilon/3 = 2\varepsilon/3 \text{ so } z \notin H.$$

Thus $G \cap H \ne \{empty\}$ and the topology is Hausdorff.

11.2) Suppose that $\{p_1, p_2, p_3, \cdots\}$ is a countable family of seminorms on a linear space X. Prove that $d(x,y) = \sum_{n \in N} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$ defines a metric on X and prove that the metric topology defined by d coincides with the one defined by the family of seminorms $\{p_1, p_2, p_3, \cdots\}$.

Defines a metric: In exercise 1.5 we showed that if p is a seminorm then

$$\frac{p(x-y)}{1+p(x-y)} \le \frac{p(x-z)}{1+p(x-z)} + \frac{p(z-y)}{1+p(z-y)}$$

So the triangle inequality holds for each individual seminorm.

For any $\varepsilon > 0$ we can pick an N s.t. $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon$. We can then say $d(x,y) \le d(x,z) + d(z,y) + \varepsilon$.

Since this holds for any $\varepsilon > 0$ we can conclude that $d(x,y) \le d(x,z) + d(z,y)$.

Topology coincides: Let T be the topology that the seminorms $\{p_1, p_2, p_3, \cdots\}$ induce.

Then $x_j \to x$ in T $\Leftrightarrow$ $p_n(x_j - x) \to 0$ for all n.

We want to show that $p_n(x_j - x) \to 0$ for all n if and only if $d(x_j, x) \to 0$.

$\Rightarrow$) Assume $p_n(x_j - x) \to 0$ and fix $\varepsilon > 0$. Pick an N s.t. $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon$.

Then $$d(x,x_j) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{p_n(x-x_j)}{1+p_n(x-x_j)} + \underbrace{\sum_{n=N+1}^{\infty} \frac{1}{2^n} \underbrace{\frac{p_n(x-x_j)}{1+p_n(x-x_j)}}_{\le 1}}_{<\varepsilon} \xrightarrow{j \to \infty} \varepsilon.$$

Note that the first sum has a finite number of terms and each term is going to zero.

We now have $\limsup_{j \to \infty} d(x,x_j) < \varepsilon$ for all $\varepsilon > 0$ which implies that $d(x_j, x) \to 0$.

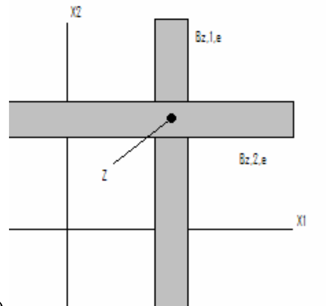
$\Leftarrow$) (By contradiction) Assume that $p_{n_0}(x_j - x) \not\to 0$ for some n_0.

Then there exist $(j_k)_{k=1}^{\infty}$ and $\varepsilon > 0$ s.t. $p_{n_0}(x_{j_k} - x) > \varepsilon$ for all k.

Then $d(x, x_{j_k}) \ge \frac{1}{2^{n_0}} \frac{p_{n_0}(x - x_{j_k})}{1 + p_{n_0}(x - x_{j_k})} > \frac{1}{2^{n_0}} \frac{\varepsilon}{1+\varepsilon}$ so $d(x_j, x) \not\to 0$.

Problem) Consider the linear space $L = R^2$ and define for $x = (x_1, x_2) \in L$ the seminorms $p_1(x) = |x_1|, p_2(x) = |x_2|$.

Construct for $x \in L, j \in \{1,2\}, \varepsilon \in (0,\infty)$ the sets $B_{x,j,\varepsilon} = \{y \in L : p_j(x-y) < \varepsilon\}$.



$B_{x,1,\varepsilon} = \{y \in L : p_1(x-y) < \varepsilon\} = \{y \in L : |x_1 - y_1| < \varepsilon\}$
$B_{x,2,\varepsilon} = \{y \in L : p_2(x-y) < \varepsilon\} = \{y \in L : |x_2 - y_2| < \varepsilon\}$

Let T denote the topology generated by $\{p_1\}$. Let T_1 denote the standard topology on R.
Then $G \in T \Leftrightarrow \exists H \in T \; s.t. \, G = H \otimes R$.
This topology is not Hausdorff because it cannot separate between distinct points with the same first coordinate (i.e. two points that lie in the same vertical line in the picture)

Set $\|x\| = p_1(x) + p_2(x)$.
Then $\|\cdot\|$ is a seminorm and the topology generated by $\{p_1, p_2\}$ is the same as the topology generated by $\|\cdot\|$.
Also, if $\|x\| = 0$ then $x_1 = x_2 = 0$ which implies that $\|\cdot\|$ is a norm.
This topology is Hausdorff because any topology generated by a norm is Hausdorff.
Finally, note that on a finite dimensional space any two norms are equivalent. We can therefore conclude that the topology defined by $\|\cdot\|$ is the standard topology on R^2.