Homework 5

9.1) Prove that $\rho(A^*) = \overline{\rho(A)}$ where $\overline{\rho(A)}$ is the set $\{\lambda \in C \mid \overline{\lambda} \in \rho(A)\}$.

Assume $\lambda \in \rho(A)$. Then $(A-\lambda I)^{-1}$ and $((A-\lambda I)^{-1})^*$ exist and are bounded.
We need to show that $((A-\lambda I)^{-1})^* = (A^* - \overline{\lambda} I)^{-1}$.
If we show that $(B^{-1})^* = (B^*)^{-1}$ then this follows immediately.

Say $y = By'$. Then
$$\left\langle (B^{-1})^* x, y \right\rangle = \left\langle x, B^{-1} y \right\rangle = \langle x, y' \rangle$$
$$\left\langle (B^*)^{-1} x, y \right\rangle = \left\langle (B^*)^{-1} x, By' \right\rangle = \left\langle B^* (B^*)^{-1} x, y' \right\rangle = \langle x, y' \rangle$$

So $(B^{-1})^* = (B^*)^{-1}$ and we are done.

9.2) If λ is an eigenvalue of A then $\overline{\lambda}$ is in the spectrum of A^*. What can you say about the type of spectrum $\overline{\lambda}$ belongs to?

First we show that $\overline{\lambda}$ is in the spectrum of A^*: $\lambda \in \sigma_p(A) \Rightarrow \exists x \neq 0 \; s.t. \; (A - \lambda I)x = 0 \; \forall y$
This holds iff $\left(x, (A^* - \overline{\lambda} I)y\right) = 0 \; \forall y$ which holds iff $x \perp ran(A^* - \overline{\lambda} I) \Rightarrow \overline{\lambda} \in \sigma(A^*)$
Now $\overline{\lambda} \notin \sigma_c(A)$ because $(A^* - \overline{\lambda} I)$ is dense iff $(A^* - \overline{\lambda} I)^{\perp} = 0$, but $x \neq 0$.
So $\overline{\lambda}$ is in either the point or residual spectrum of A^*.

9.3) Suppose that A is a bounded linear operator of a Hilbert space and $\lambda, \mu \in \rho(A)$. Prove that the resolvent R_λ of A satisfies $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$.

First note that $A^{-1} - B^{-1} = -A^{-1}(A - B)B^{-1}$ (we use this below in the equality denoted by *)

Then $R_\lambda - R_\mu = (A - \lambda I)^{-1} - (A - \mu I)^{-1} \overset{*}{=} -\underbrace{(A - \lambda I)^{-1}}_{=R_\lambda}\underbrace{((A - \lambda I) - (A - \mu I))}_{=(\mu - \lambda)}\underbrace{(A - \mu I)^{-1}}_{=R_\mu} = (\mu - \lambda) R_\lambda R_\mu$

9.4) Prove that the spectrum of an orthogonal projection P is either $\{0\}$ (in which case P=0), $\{1\}$ (in which case P=I), or $\{0,1\}$.

Assume that P is an orthogonal projection. Then $H = ranP \oplus \ker P$ where $ranP = (\ker P)^\perp$.

Case 1) $ranP = \{0\}$

Then $P = 0$ and $Px = 0 \ \forall x$ so $0 \in \sigma_p(P)$

If $\lambda \neq 0$ then $(P - \lambda I)^{-1} = \frac{1}{\lambda} I$ so $\lambda \in \rho(P)$

Case 2) $\ker P = \{0\}$

Then $ranP = (\ker P)^\perp = H$ so $P = I$ and $Px = x \ \forall x$ so $1 \in \sigma_p(P)$

If $\lambda \neq 1$ then $(P - \lambda I)^{-1} = \frac{1}{1-\lambda} I$ so $\lambda \in \rho(P)$

Case 3) $ranP \neq \{0\}, \ker P \neq \{0\}$

If $x \neq 0, x \in ranP$ then $x = Px$ so $1 \in \sigma_p(P)$

If $x \neq 0, x \in \ker P$ then $0 = Px$ so $0 \in \sigma_p(P)$

If $\lambda \neq 0,1$ then $(P - \lambda I)^{-1} = \frac{1}{1-\lambda} P - \frac{1}{\lambda}(I - P)$ so $\lambda \in \rho(P)$

9.5) A is a bounded, nonnegative operator on a complex Hilbert space. Prove that $\sigma(A) \subset [0, \infty)$.

First note that A nonnegative implies A self-adjoint and A self-adjoint implies $\sigma(A) \in R$. Also, A bounded implies $\sigma(A) \subseteq [-\|A\|, \|A\|]$.

Assume $\lambda < 0$. We need to show that $(A - \lambda I)$ is invertible.

Since A is self-adjoint we know that $(Au, u) = (u, Au) \in R$ so:

$\|(A - \lambda I)u\|^2 = \underbrace{\|Au\|^2}_{\geq 0} \underbrace{- 2\underbrace{\lambda}_{<0}\underbrace{(Au, u)}_{\geq 0}}_{\geq 0} + \lambda^2 \|u\|^2 \geq \lambda^2 \|u\|^2$ so $(A - \lambda I)$ is coercive. A coercive implies

$\begin{cases} ran(A - \lambda I)\, closed \Rightarrow \lambda \notin \sigma_c(A) \\ (A - \lambda I)\, one-to-one \Rightarrow \lambda \notin \sigma_p(A) \end{cases}$. A self-adjoint implies $\sigma_r(A) = \{empty\}$.

Since λ is not in any of the parts of the spectrum it is not in the spectrum and our proof is complete.

9.6) G is a multiplication operator on $L^2(R)$ defined by $Gf(x)=g(x)f(x)$ where g is continuous and bounded. Prove that G is a bounded linear operator on $L^2(R)$ and that its spectrum is given by $\sigma(G)=\overline{\{g(x)\mid x\in R\}}$. Can an operator of this form have eigenvalues?

G is a bounded linear operator:

$$\|G\|=\sup_{\|f\|=1}\|Gf\|=\sup_{\|f\|=1}\left(\int|g(x)f(x)|^2dx\right)^{1/2}\le\underbrace{\sup|g(x)|}_{=\|g\|_u}\underbrace{\sup_{\|f\|=1}\left(\int|f(x)|^2dx\right)^{1/2}}_{=\|f\|=1}=\|g\|_u$$

Spectrum: Set $\Omega=\overline{\{g(x)\mid x\in R\}}$.

Suppose $\lambda\notin\Omega$. Then $\exists\varepsilon>0$ s.t. $|\lambda-g(x)|\ge\varepsilon\ \forall x$.

Note that

$$(G-\lambda I)\frac{1}{g(x)-\lambda}f(x)=\frac{g(x)}{g(x)-\lambda}f(x)-\frac{\lambda}{g(x)-\lambda}f(x)=f(x)\Rightarrow(G-\lambda I)^{-1}f(x)=\frac{1}{g(x)-\lambda}f(x)$$

Then

$$\left\|(G-\lambda I)^{-1}\right\|=\sup_{\|f\|=1}\left\|(G-\lambda I)^{-1}f\right\|=\sup_{\|f\|=1}\left(\int\left|\frac{1}{g(x)-\lambda}f(x)\right|^2dx\right)^{1/2}\le\underbrace{\sup\left|\frac{1}{g(x)-\lambda}\right|}_{\le\varepsilon}\underbrace{\sup_{\|f\|=1}\left(\int|f(x)|^2dx\right)^{1/2}}_{=\|f\|=1}\le\varepsilon$$

Suppose $\lambda\in\Omega$. Then there exists $x_n\in R$ s.t. $g(x_n)\to\lambda$

For $j=1,2,3...$ pick n_j s.t. $\left|g(x_{n_j})-\lambda\right|\le\frac{1}{j}$

Since g is continuous at x_{n_j} there exists δ s.t. $x\in B_\delta(x_{n_j})\Rightarrow\left|g(x_{n_j})-g(x)\right|\le\frac{1}{j}$

Set $u_{n_j}(x)=\begin{cases}\sqrt{j/2} & x\in B_\delta(x_{n_j})\\ 0 & else\end{cases}$

Then

$$\left\|(G-\lambda I)u_{n_j}\right\|^2=\int|g(x)-\lambda|^2\left|u_{n_j}(x)\right|^2dx\overset{TI}{\le}\int\left(\underbrace{\left|g(x)-g(x_{n_j})\right|}_{\le1/j}+\underbrace{\left|g(x_{n_j})-\lambda\right|}_{\le1/j}\right)^2\left|u_{n_j}(x)\right|^2dx\le$$

$$\le\frac{4}{j^2}\underbrace{\int\left|u_{n_j}(x)\right|^2dx}_{=1}=\frac{4}{j^2}\xrightarrow{j\to\infty}0$$

The inequality denoted by "TI" uses the triangle inequality.

We have shown that $(G-\lambda I)$ is not continuously invertible (so λ is in the spectrum).

Eigenvalues: Suppose $(G-\lambda I)u=0$ for $u\ne0$.

Then $(g(x)-\lambda)u(x)=0$ but $u\ne0$. This is possible if and only if the set $\{x:g(x)=\lambda\}$ has positive (non-zero) measure).

9.7) Let $K: L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by $Kf(x) = \int_0^x f(y)dy$.

a) Find the adjiont operator K^*.

$$(Kf,g) = \int_0^1 \int_0^x \overline{f(y)}dy\, g(x)dx = \int_0^1 \int_0^x \overline{f(y)} g(x)dydx = \int_0^1 \int_y^1 \overline{f(y)} g(x)dxdy = \int_0^1 \overline{f(y)} \int_y^1 g(x)dxdy = (f, K^* g)$$

So $K^* g(x) = \int_y^1 g(y)dy$

b) Show that $\|K\| = 2/\pi$.

Set $\phi_n(x) = \sqrt{2}\cos\left(\frac{n\pi x}{2}\right)$. Then $(\phi_n)_{n=1}^\infty$ is an ON-basis for $L^2([0,1])$.

Then $[K\phi_n](x) = \sqrt{2}\int_0^x \cos\left(\frac{n\pi x}{2}\right)dy = \left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^x = \sqrt{2}\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$.

Set $\psi_n(x) = \sqrt{2}\sin\left(\frac{n\pi x}{2}\right)$. Then $(\psi_n)_{n=1}^\infty$ is also an ON-basis for $L^2([0,1])$.

We can write $x = \sum_{n=1}^\infty \alpha_n \phi_n$.

Then $\|Kx\|^2 = \left\|\sum_{n=1}^\infty \alpha_n K\phi_n\right\|^2 = \left\|\sum_{n=1}^\infty \alpha_n \frac{2}{n\pi}\psi_n\right\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 \underbrace{\left(\frac{2}{n\pi}\right)^2}_{\le 4/\pi^2} \le \frac{4}{\pi^2}\sum_{n=1}^\infty |\alpha_n|^2 = \frac{4}{\pi^2}\|x\|^2$ so $\|K\| \le \frac{2}{\pi}$.

Since $\|K\phi_1\|^2 = \frac{2}{\pi}\|\phi_1\|^2$ we also have that $\|K\| \ge \frac{2}{\pi}$. Together we get $\|K\| = 2/\pi$.

Remark: We have determined the singular value decomposition of K:

$Kx = \sum_{n=1}^\infty \sigma_n \psi_n \langle \phi_n, x\rangle$ where $\sigma_n = \frac{2}{n\pi}$ are the singular values.

We can then conclude that $\|K\| = \max_n \sigma_n = \sigma_1 = \frac{2}{\pi}$.

c) Show that the spectral radius of K is equal to zero.

$$[K^2u](x) = \int_0^x\int_0^y u(z)dzdy = \int_0^x u(z)\int_z^x dydz = \int_0^x (x-z)u(z)dz$$

$$[K^3u](x) = \int_0^x\int_0^y (y-z)u(z)dzdy = \int_0^x u(z)\int_z^x (y-z)dydz = \int_0^x \frac{(x-z)^2}{2}u(z)dz$$

This generalizes to $[K^n u](x) = \cdots = \int_0^x \frac{(x-z)^{n-1}}{(n-1)!}u(z)dz$

So $\|K^n u\|^2 = \int_0^1\left(\int_0^x \frac{(x-z)^{n-1}}{(n-1)!}u(z)dz\right)^2 dx \underset{CS}{\leq} \frac{1}{(n-1)!}\int_0^1 \underbrace{\left(\int_0^x (x-z)^{2(n-1)}dz\right)^2}_{\leq 1}\underbrace{\left(\int_0^x u^2(y)dz\right)^2}_{\leq \|u\|^2} dx \leq \frac{\|u\|^2}{(n-1)!}$

This implies that $\|K^n\|^2 \leq \frac{1}{(n-1)!}$, so $r(K) = \limsup_{n\to\infty}\left(\frac{1}{(n-1)!}\right)^{1/n} = 0$

d) Show that 0 belongs to the continuous spectrum of K.

Set $Ku = v$.

Pick $\tilde{v} \in P$ s.t. $\|v - \tilde{v}\| < \varepsilon$ where P is the set of functions $(\sin(n\pi x))_{n=1}^{\infty}$. We have previously shown that this is a basis.

Set $\tilde{u} = \tilde{v}'$ then $\tilde{u} \in L^2(I)$ and $[K\tilde{u}](x) = \int_0^x \tilde{v}(y)dy = \tilde{v}(x) - \tilde{v}(0) = \tilde{v}(x)$

The final equality uses the fact that $\sin(0) = 0$.

So $\tilde{v}(x) \in ranK$ and $\|v - \tilde{v}\| < \varepsilon$, hence 0 is in the continuous spectrum.

9.8) Define the right shift operator S on $l^2(Z)$ by $S(x)_k = x_{k-1}\ \forall k \in Z$ where $x = (x_k)_{k=-\infty}^{\infty}$ is in $l^2(Z)$. Prove the following (a-d).

First recall the Fourier transform: $F^{-1}x = \sum_{n=-\infty}^{\infty} x_n \frac{e^{itn}}{\sqrt{2\pi}}$

Set $\widetilde{S} = F^{-1}SF$, then $(\widetilde{S} - \lambda I) = F^{-1}SF - \lambda F^{-1}F = F^{-1}(S - \lambda I)F$

Now $\lambda \in \sigma_\alpha(S) \Leftrightarrow \lambda \in \sigma_\alpha(\widetilde{S}),\quad \alpha = p, c, r$

Then $F^{-1}Sx = \sum_{n=-\infty}^{\infty} x_{n-1} \frac{e^{itn}}{\sqrt{2\pi}} = e^{it} \sum_{n=-\infty}^{\infty} x_{n-1} \frac{e^{it(n-1)}}{\sqrt{2\pi}} = e^{it} \underbrace{\widetilde{x}(t)}_{=F^{-1}x}$ so $[\widetilde{S}\widetilde{x}](t) = e^{it}\widetilde{x}(t)$

Assume $|\lambda| \neq 1$. Given $\widetilde{y} \in L^2(T)$ we have $(S - \lambda I)\frac{1}{e^{it} - \lambda}\widetilde{y}(t) = \widetilde{y}(t)$ so $(S - \lambda I)$ is bijective.

$(S - \lambda I)$ bijective implies $\lambda \in \rho(\widetilde{S})$ (*)

Assume $|\lambda| = 1$ and $(\widetilde{S} - \lambda I)\widetilde{x} = 0$. Then $(e^{it} - \lambda I)\widetilde{x}(t) = 0$ almost everywhere which implies $\widetilde{x} = 0$.

This means that $(\widetilde{S} - \lambda I)$ is one-to-one, so we can immediately conclude that $\lambda \notin \sigma_p(\widetilde{S})$. (**)

Note that $1 \notin ran(\widetilde{S} - \lambda I) \Rightarrow ran(\widetilde{S} - \lambda I) \neq L^2(T)$ (***)

However, given a $\widetilde{y} \in L^2(T)$ set $\widetilde{y}_m(t) = \begin{cases} \widetilde{y}(t) & |\lambda - e^{it}| \geq 1/n \\ 0 & else \end{cases}$ then $\widetilde{y}_m(t) \xrightarrow{m \to \infty} y(t)$ and

$\widetilde{y}_m \in ran(\widetilde{S} - \lambda I)$ since $(\widetilde{S} - \lambda I)\frac{\widetilde{y}_m(t)}{e^{it} - \lambda} = \widetilde{y}_m(t)$ (****)

a) The point spectrum of S is empty.

The equations (*) and (**) above show that λ isn't in the point spectrum for $|\lambda| \neq 1$ and $|\lambda| = 1$ respectively. Combined they show that the point spectrum is empty.

b) $ran(\lambda I - S) = l^2(Z)$ for every $\lambda \in C$ with $|\lambda| > 1$

Equation (*) above shows this.

c) $ran(\lambda I - S) = l^2(Z)$ for every $\lambda \in C$ with $|\lambda| < 1$

Equation (*) above shows this.

d) The spectrum of S consists of the unit circle $\{\lambda \in C \mid |\lambda| = 1\}$ and is purely continuous.

Equation (*) shows that λ is not in the spectrum for $|\lambda| \neq 1$. Equations (***) and (****) combine to show that all λ with $|\lambda| = 1$ are in the continuous spectrum.

9.9) Define the discrete Laplacian operator Δ on $l^2(Z)$ by $(\Delta x)_k = x_{k-1} - 2x_k + x_{k+1}$, where $x = (x_k)_{k=-\infty}^{\infty}$. Show that $\Delta = S + S^* - 2I$ and prove that the spectrum of Δ is entirely continuous and consists of the interval $[-4,0]$.

Noting that on $l^2(Z)$ the adjoint of the right shift operator is the left shift operator (see problem 3 of homework 3), the fact that $\Delta = S + S^* - 2I$ follows directly.

Spectrum: As we did in the previous problem we begin by switching to the Fourier domain. Then

$$F^{-1}\Delta x = \sum_{n=-\infty}^{\infty}(x_{n-1} + x_{n+1} + 2x_n)\frac{e^{itn}}{\sqrt{2\pi}} = e^{it}\sum_{n=-\infty}^{\infty} x_{n-1}\frac{e^{it(n-1)}}{\sqrt{2\pi}} + e^{-it}\sum_{n=-\infty}^{\infty} x_{n+1}\frac{e^{it(n+1)}}{\sqrt{2\pi}} + 2\sum_{n=-\infty}^{\infty} x_n\frac{e^{itn}}{\sqrt{2\pi}} = (e^{it} + e^{-it} + 2)\underbrace{\tilde{x}(t)}_{=f^{-1}x}$$

so $[\tilde{\Delta}\tilde{x}](t) = (e^{it} + e^{-it} + 2)\tilde{x}(t)$

Note that $e^{it} + e^{-it} + 2 \le \sup|e^{it}| + \sup|e^{-it}| + 2 \le 4$

Assume $|\lambda| > 4$. Given $\tilde{y} \in L^2(T)$ we have $(\Delta - \lambda I)\frac{1}{(e^{it} + e^{-it} + 2) - \lambda}\tilde{y}(t) = \tilde{y}(t)$ so $(\Delta - \lambda I)$ is bijective.

Note that $e^{it} + e^{-it} + 2 \ge -\sup|e^{it}| - \sup|e^{-it}| + 2 \ge 0$

Assume $|\lambda| < 4$. Given $\tilde{y} \in L^2(T)$ we have $(\Delta - \lambda I)\frac{1}{(e^{it} + e^{-it} + 2) - \lambda}\tilde{y}(t) = \tilde{y}(t)$ so $(\Delta - \lambda I)$ is bijective.

So the spectrum consists of the interval $[-4,0]$. We just need to show that it is continuous.

Continuous: Note that $1 \notin ran(\tilde{\Delta} - \lambda I) \Rightarrow ran(\tilde{\Delta} - \lambda I) \ne L^2(T)$

However, given a $\tilde{y} \in L^2(T)$ set $\tilde{y}_m(t) = \begin{cases} \tilde{y}(t) & |\lambda - (e^{it} + e^{-it} + 2)| \ge 1/n \\ 0 & else \end{cases}$ then $\tilde{y}_m(t) \xrightarrow{m\to\infty} y(t)$

and $\tilde{y}_m \in ran(\tilde{\Delta} - \lambda I)$ since $(\tilde{\Delta} - \lambda I)\frac{\tilde{y}_m(t)}{(e^{it} + e^{-it} + 2) - \lambda} = \tilde{y}_m(t)$

9.10) Posted separately on the website.

9.11) The approximate spectrum is defined $\sigma_{app}(A) = \{\lambda : \exists (x_n) \; s.t. \; \|x_n\| = 1 \, and \; \|(A - \lambda I)x_n\| \to 0\}$.
Prove the following: (a) $\sigma_{app}(A) \subseteq \sigma(A)$

(b) $\sigma_p(A) \subseteq \sigma_{app}(A)$

(c) $\sigma_c(A) \subseteq \sigma_{app}(A)$

(d) Give an example to show that a point in the residual spectrum need not belong to the approximate spectrum.

a) Prove $\sigma_{app}(A) \subseteq \sigma(A)$

Assume $\lambda \in \sigma(A)^c = \rho(A)$. Then $(A - \lambda I)^{-1}$ is a bounded operator. If (x_n) is any sequence of vectors with $\|x_n\| = 1$ then set $y_n = (A - \lambda I)x_n$.

Then $1 = \|x_n\| = \|(A - \lambda I)^{-1} y_n\| \le \|(A - \lambda I)^{-1}\| \cdot \|y_n\|$.

Also $\|y_n\| = \|(A - \lambda I)x_n\| \ge \dfrac{1}{\|(A - \lambda I)^{-1}\|}$ so $\lambda \notin \sigma_{app}(A)$.

b) Prove $\sigma_p(A) \subseteq \sigma_{app}(A)$

Assume $\lambda \in \sigma_p(A)$. Then there exists an $x \ne 0$ s.t. $Ax = \lambda x$. Set $x_n = \dfrac{x}{\|x\|}$, then $\|(A - \lambda I)x_n\| = 0$ so $\lambda \in \sigma_{app}(A)$.

c) Prove $\sigma_c(A) \subseteq \sigma_{app}(A)$

Assume $\lambda \in \sigma_c(A)$. Then $\overline{ran(A - \lambda I)} = H$. Set $\alpha = \inf_{\|x\|=1} \|(A - \lambda I)x\|$. We want to prove that $\alpha = 0$ (if it is then we can pick x_n s.t. $\|x_n\| = 1 \, and \; \|(A - \lambda I)x_n\| \to 0$).

If $\alpha \ne 0$ then by Proposition 5.30 $ran(A - \lambda I)$ is closed. This is impossible since $(A - \lambda I)$ is not onto but $\overline{ran(A - \lambda I)} = H$.

d) Give an example of an operator A and a point $\lambda \in \sigma_r(A)$ s.t. $\lambda \notin \sigma_{app}(A)$.

Consider the right-shift operator S from question 9.10 and the point $\lambda = 0$. Then if $\|x_n\| = 1$ we have $\|(S - \lambda I)x_n\| = \|Sx_n\| = \|x_n\| = 1$.