**Homework set 2 — APPM5450 Spring 2011 — Solutions:**

**Exercise 7.13:** Set $I = [0,\,1]$ and consider the equation

$$i\,u_t = -u_{xx}, \qquad x \in I, \qquad t > 0, \tag{1}$$

for a complex valued function $u = u(x,t)$ with homogeneous boundary conditions,

$$u(0,t) = u(1,t) = 0, \tag{2}$$

and initial condition

$$u(x,0) = f(x). \tag{3}$$

Set

$$e_n(x) = \sqrt{2}\,\sin(n\,x).$$

Then $(e_n)_{n=1}^{\infty}$ forms an ON-basis for $L^2(I)$. We look for a solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\,e_n(x). \tag{4}$$

Inserting (4) into (1) and (3), we find that $\alpha_n$ must satisfy

$$i\,\alpha_n' = n^2\,\alpha_n, \qquad \alpha_n(0) = f_n,$$

where $f_n = (e_n, f)$. The solution is

$$\alpha_n(t) = f_n\,e^{-i\,n^2\,t}.$$

Since $|\alpha_n(t)| = |f_n|$ for any $t$, it follows directly from Parseval that

$$||u(t)||_{L^2(I)}^2 = \sum_{n=1}^{\infty} |\alpha_n(t)|^2 = \sum_{n=1}^{\infty} |f_n|^2 = ||f||^2,$$

and that (using that the cosines also form an ON-set)

$$||u_x(t)||_{L^2(I)}^2 = ||\sum_{n=1}^{\infty} f_n\,e^{-i\,n^2\,t}\,n\,\sqrt{2}\cos(nx)||_{L^2(I)}^2 = \sum_{n=1}^{\infty} |n\,f_n|^2 = ||f_x||^2.$$

For a direct proof, set $v = \mathrm{Re}(u)$ and $w = \mathrm{Im}(u)$ so that $u = v + i\,w$. Then (1) takes the form

$$v_t = -w_{xx} \qquad w_t = v_{xx}.$$

Now

$$\begin{aligned}\frac{d}{dt}\int_0^1 |u|^2\,dx = \frac{d}{dt}\int_0^1 (v^2 + w^2)\,dx &= 2\int_0^1 (v_t\,v + w_t\,w)\,dx \\ &= 2\int_0^1 (-w_{xx}\,v + v_{xx}\,w)\,dx = 2\int_0^1 (w_x\,v_x - v_x\,w_x)\,dx = 0.\end{aligned}$$

The second to last step was partial integration where the boundary terms vanish due to (2). Analogously,

$$\begin{aligned}\frac{d}{dt}\int_0^1 |u_x|^2\,dx = \frac{d}{dt}\int_0^1 (v_x^2 + w_x^2)\,dx &= 2\int_0^1 (v_{xt}\,v_x + w_{xt}\,w_x)\,dx \\ &= 2\int_0^1 (-v_t\,v_{xx} - w_t\,w_{xx})\,dx = 2\int_0^1 (-v_t\,w_t + w_t\,v_t)\,dx = 0.\end{aligned}$$

In the second calculation we used differentiation, (2) takes the form

$$v_t(0,t) = v_t(1,t) = w_t(0,t) = w_t(1,t) = 0, \qquad t > 0.$$

**Exercise 8.3:** Let $P$ and $Q$ be orthogonal projections. Set $M = \operatorname{ran}(P)$ and $N = \operatorname{ran}(Q)$. TFAE:

(a) $M \subseteq N$
(b) $QP = P$
(c) $PQ = P$
(d) $||Px|| \leq ||Qx|| \qquad \forall x$
(e) $(x, Px) \leq (x, Qx) \qquad \forall x$

*Proof:*

(a) $\Rightarrow$ (b): Assume $M \subseteq N$. Then for any $x$, $Px \in M \subseteq N$, so $QPx = Px$.

(b) $\Rightarrow$ (a): Assume $QP = P$. Pick $y \in M$. Then $y = Px$ for some $x$. Then $Qy = QPx = Px = y$ so $y \in N$.

(a) $\Leftrightarrow$ (c):

$$\begin{aligned} M \subseteq N \quad &\Leftrightarrow \quad N^\perp \subseteq M^\perp \\ &\Leftrightarrow \quad Py = 0 \quad \forall y \in N^\perp \\ &\Leftrightarrow \quad P(I-Q)x = 0 \quad \forall x \\ &\Leftrightarrow \quad P = PQ \end{aligned}$$

(c) $\Rightarrow$ (d): Assume $PQ = P$. Since $||P|| \leq 1$ we have $||Px|| = ||PQx|| \leq ||Qx||$ for any $x$.

(d) $\Rightarrow$ (a): Assume that (a) is false. Then there is an $x \in M \backslash N$. Since $x \in M$ we have $x = Px$ and so

$$||Px||^2 = ||x||^2 = ||Qx + (I-Q)x||^2 = ||Qx||^2 + ||(I-Q)x||^2.$$

Now observe that $||(I-Q)x|| > 0$ since $x \notin N$. Consequently,

$$||Qx||^2 = ||Px||^2 - ||(I-Q)x||^2 < ||Px||^2$$

so (d) cannot hold true.

(d) $\Leftrightarrow$ (e): Simply observe that $(x, Px) = (x, P^2x) = (Px, Px) = ||Px||^2$ and analogously $(x, Qx) = ||Qx||^2$.

*Note:* You may want to draw a diagram over the implications to convince yourself that all equivalencies have been proven.

**Exercise 8.4:** First we prove that $P_n \to I$ strongly. Fix any $x \in H$. Since $\bigcup_{n=1}^{\infty} \mathrm{ran}(P_n) = H$, we know that $x \in \mathrm{ran}(P_N)$ for some specific $N$. Then, since $\mathrm{ran}(P_n) \subseteq \mathrm{ran}(P_{n+1})$, we see that $x \in \mathrm{ran}(P_m)$ for any $m \geq N$. Consequently, $P_m x = x$ for any $m \geq N$ so $P_n x \to x$ (*very* rapidly!).

Next suppose that $||I - P_n|| \to 0$. Then there is some $N$ such that $||I - P_N|| \leq 1/2$. Now observe that $I - P_N$ is itself an orthogonal projection (onto $\ker(P_N)$) so it can only have norms 0 and 1. It follows that $||I - P_N|| = 0$, which is to say that $P_N = I$. Since $H = \mathrm{ran}(P_N) \subseteq \mathrm{ran}(P_{N+1}) \subseteq \mathrm{ran}(P_{N+2}) \subseteq \cdots$ we see that $P_n = I$ for any $n \geq N$.

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**Problem 1:** Let $T(t)$ denote the semigroup defined in Section 7.3 of the textbook. Prove that $T(t) \to I$ strongly as $t \searrow 0$. Prove that $T(t)$ does not converge in norm.

*Solution:* We consider a slightly more general problem. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for a Hilbert space $H$, and consider for $t \geq 0$ the operator

$$T(t)f = \sum_{n=1}^{\infty} f_n \, e^{-n^2 t} \, e_n.$$

We will show that as $t \searrow 0$, $T(t) \to I$ strongly but not in norm.

To show $T(t) \to I$ strongly, fix $f \in H$. Fix $\varepsilon > 0$. Set $f_n = (e_n, f)$ and pick $N$ such that $\sum_{n=N+1}^{\infty} |f_n|^2 < \varepsilon^2$. Then by Parseval

$$\begin{aligned} ||T(t)f - f||^2 &= \sum_{n=1}^{N} \Big| f_n \, (e^{-n^2 t} - 1) \Big|^2 + \sum_{n=N+1}^{\infty} \Big| f_n \, (e^{-n^2 t} - 1) \Big|^2 \\ &\leq \sum_{n=1}^{N} \Big| f_n \, (e^{-n^2 t} - 1) \Big|^2 + \sum_{n=N+1}^{\infty} 4 \, |f_n|^2 \leq \sum_{n=1}^{N} \Big| f_n \, (e^{-n^2 t} - 1) \Big|^2 + 4\varepsilon^2. \end{aligned}$$

Since only finitely many terms depend on $t$, we can now easily take the limit as $t \searrow 0$,

$$\limsup_{t \searrow 0} ||T(t)f - f||^2 \leq 4 \, \varepsilon^2.$$

Since $\varepsilon$ was arbitrary, we see that $\lim_{t \searrow 0} ||T(t)f - f|| = 0$.

To show that $T(t)$ does not converge to $I$ in norm, we simply observe that for any $t > 0$

$$||T(t) - I|| \geq \sup_n ||\big(T(t) - I\big) \, e_n|| = \sup_n |e^{-n^2 t} - 1| = 1.$$

**Problem 2:** Suppose $P$ is a projection on a Hilbert space $H$. TFAE:

(a) $P$ is orthogonal, *i.e.* $\ker(P) = \mathrm{ran}(P)^{\perp}$.

(b) $P$ is self-adjoint, *i.e.* $\langle P\,x,\, y\rangle = \langle x,\, P\,y\rangle \quad \forall x, y$.

(c) $||P|| = 0$ or $1$.

*Proof:*

(a) $\Rightarrow$ (b): Assume $\ker(P) = \mathrm{ran}(P)^{\perp}$. Pick any $x, y \in H$. Then

$$(Px,\, y) = (\underbrace{Px}_{\in\mathrm{ran}(P)},\, Py + \underbrace{(I-P)y}_{\in\ker(P)}) = (Px,\, Py) = (Px + (I-P)x,\, Py) = (x,\, Py).$$

(b) $\Rightarrow$ (c): Assume that (b) holds. Then for any $x$,

$$||Px||^2 = (Px,\, Px) = (P^2x,\, x) = (Px,\, x) \leq ||Px||\,||x||,$$

so $||P|| \leq 1$. Obviously it is possible for $||P||$ to be zero. We need to prove that the only possible non-zero value of $||P||$ is one. To this end, note that if $P \neq 0$, then $\mathrm{ran}(P) \neq \{0\}$. Now observe that if $x$ is a non-zero element in $\mathrm{ran}(P)$, we have $Px = x$ so $||P|| \geq 1$.

(c) $\Rightarrow$ (a): Assume that (a) does not hold. Then there exist $x \in \mathrm{ran}(P)$ and $y \in \ker(P)$ such that $(x,\, y) \neq 0$. Set $\alpha = \overline{(x,\, y)}/|(x,\, y)|$ and $z = \alpha y$. Then $z \in \ker(P)$ and $(x,\, z) = |(x,\, y)| \in \mathbb{R}_+$. Set

$$w = x - z\,t.$$

Then $||Pw|| = ||x||$, and

$$||w||^2 = ||x||^2 - 2\,t\,(x,\, z) + t^2\,||z||^2.$$

For small $t$, we see that $||w|| < ||x|| = ||Pw||$ so $||P|| > 1$.

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*No solution is given for Problem 3 since the problem itself outlines precisely how to solve it — just fill in the details.*