**Homework set 14 — APPM5450, Spring 2011**

**From the book:** 12.8, 12.16, 12.17, 12.18. Optional: 12.13, 12.14, 12.15.

**Problem 1:** Let $\lambda$ be a real number such that $\lambda \in (0, 1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that

$$a^{\lambda}\, b^{1-\lambda} \leq \lambda\, a + (1-\lambda)\, b, \tag{1}$$

with equality iff $a = b$.

*Hint:* Consider the case $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

**Problem 2:** [Hölder's inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^P(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

$$||fg||_1 \leq ||f||_p\, ||g||_q. \tag{2}$$

Prove that equality holds iff $\alpha |f|^p = \beta |g|^q$ for some $\alpha, \beta$ such that $\alpha\beta \neq 1$.

*Hint:* Consider first the case where $||f||_p = 0$ or $||g||_q = 0$. For the case $||f||_p\, ||g||_q \neq 0$, use (1) with

$$a = \left|\frac{f(x)}{||f||_p}\right|^p, \qquad b = \left|\frac{g(x)}{||g||_q}\right|^q, \qquad \lambda = \frac{1}{p}.$$

**Problem 3:** [Minkowski's inequality] Let $(X, \mu)$ be a measure space, and let $p$ be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

*Hint:* Consider the cases $p = 1, \infty$ separately. For $p \in (1, \infty)$, note that

$$|f(x) + g(x)|^p \leq \big(|f(x)| + |g(x)|\big)\, |f(x) + g(x)|^{p-1}, \qquad \forall\, x \in X. \tag{3}$$

Then integrate both sides of (3) and apply (2) to the right hand side.