Applied Analysis (APPM 5450): Final exam
1.30pm – 4.00pm, May 3, 2011. Closed books.

Problem 1: (14p) Let d be a positive integer denoting dimension.

(a) (4p) State the definition of the Sobolev space $H^s(\mathbb{R}^d)$ for $s \geq 0$.

(b) (4p) State the Riemann-Lebesgue lemma. (You do not need to give the proof.)

(c) (6p) Prove that if s is large enough (how large?), then $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$.

Solution:

(c) Suppose $f \in H^s$. Since $f = \mathcal{F}^*\hat{f}$, the R-L lemma states that $f \in C_0$, whenever $\hat{f} \in L^1$.

To show that $\hat{f} \in L^1$, we use the Cauchy-Schwartz inequality:

$$\int |\hat{f}| = \int (1+|t|^2)^{-s/2}\,(1+|t|^2)^{s/2}\,|\hat{f}(t)| \leq \underbrace{\left(\int (1+|t|^2)^{-s}\right)^{1/2}}_{=:C_s}\underbrace{\left(\int (1+|t|^2)^{s}\,|\hat{f}(t)|^2\right)^{1/2}}_{=||f||_{H^s}}.$$

It follows that $\int |\hat{f}| < \infty$ if $C_s < \infty$, and to determine when this happens, we switch to polar coordinates:

$$C_s = \int_{\mathbb{R}^d} (1+|t|^2)^{-s} dt = A_d \int_0^\infty (1+r^2)^{-s}\, r^{d-1}\, dr,$$

where A_d is the surface area of the unit sphere in $\mathbb{R}^d$. We see that $C_s < \infty$ iff $-2s + d - 1 < -1$, which is to say if $s > d/2$.

$\boxed{\textbf{Answer: } H^s \subseteq C_0 \text{ if } s > d/2.}$

Problem 2: (26p) Consider the Hilbert spaces $H_1 = \ell^2(\mathbb{Z})$ and $H_2 = L^2(\mathbb{R})$, and define operators $A_1 \in \mathcal{B}(H_1)$ and $A_2 \in \mathcal{B}(H_2)$ via

$$\begin{aligned} [A_1 u](n) &= \arctan(n)\, u(n) \qquad && n \in \mathbb{Z}, \\ [A_2 u](x) &= \arctan(x)\, u(x) \qquad && x \in \mathbb{R}. \end{aligned}$$

(a) (7p) Is A_1 compact? Self-adjoint? Unitary? One-to-one? Does it have closed range? Please motivate your answers briefly.

(b) (6p) Specify $\sigma(A_1)$, $\sigma_{\rm p}(A_1)$, $\sigma_{\rm c}(A_1)$, $\sigma_{\rm r}(A_1)$, and $||A_1||$. No motivation required.

(c) (7p) Is A_2 compact? Self-adjoint? Unitary? One-to-one? Does it have closed range? Please motivate your answers briefly.

(d) (6p) Specify $\sigma(A_2)$, $\sigma_{\rm p}(A_2)$, $\sigma_{\rm c}(A_2)$, $\sigma_{\rm r}(A_2)$, and $||A_2||$. No motivation required.

Solution: Let $(e_n)_{n\in\mathbb{Z}}$ denote the canonical basis for ℓ^2, and set $f_n = \chi_{(n-1/2,n+1/2)} \in H_2$.

(a) A_1 is not compact. To see this, observe that $(e_n) \rightharpoonup 0$, but $||A_1\, e_n|| \to \pi/2$.

That A_1 is self-adjoint is a trivial calculation. For $u, v \in H_1$, we find

$$(A_1 u,\, v) = \sum_{n\in\mathbb{Z}} \big(\overline{\arctan(n) u(n)}\big)\, v(n) = \sum_{n\in\mathbb{Z}} \overline{u(n)}\, \big(\arctan(n) v(n)\big) = (u,\, A_1 v).$$

A_1 is not unitary or one-to-one since, e.g., $A_1\, e_0 = 0$.

$\mathrm{ran}(A_1) = \mathrm{Span}(e_0)^\perp$ which is a closed set.

(b) $\sigma_{\rm p}(A_1) = \{\arctan(n)\}_{n\in\mathbb{Z}}$, $\sigma_{\rm c}(A_1) = \{-\pi/2, \pi/2\}$, $\sigma_{\rm r}(A_1) = \emptyset$, and $||A_1|| = \pi/2$.

Motivation: For $n \in \mathbb{Z}$, we have $A_1 e_n = \arctan(n)\, e_n$ so $\arctan(n) \in \sigma_{\rm p}(A_1)$. The points $\pm\pi/2 \in \sigma(A_1)$ since $\sigma(A_1)$ is closed, but $A_1 \pm (\pi/2)I$ is one-to-one so $\pm\pi/2 \notin \sigma_{\rm p}(A_1)$. $\sigma_{\rm r}(A_1) = \emptyset$ since A_1 is self-adjoint. Finally observe that if $\lambda \neq \arctan n$ and $\lambda \neq \pm\pi/2$, then $A_1 - \lambda I$ is invertible.

(c) A_2 is not compact. To see this, observe that $f_n \rightharpoonup 0$, but $||A_2\, f_n|| \to \pi/2$.

That A_2 is self-adjoint is a trivial calculation. For $u, v \in H_2$, we find

$$(A_2 u,\, v) = \int_{\mathbb{R}} \big(\overline{\arctan(t) u(t)}\big)\, v(t) = \int_{\mathbb{R}} \overline{u(t)}\, \big(\arctan(t) v(t)\big) = (u,\, A_2 v).$$

A_2 is not unitary since, e.g., $||A_2\, f_0|| \leq \arctan(1/2) ||f_0|| < ||f_0||$.

A_2 is one-to-one, since if $A_2 u = 0$, then $\arctan(x) u(x) = 0$ a.e. which implies that $u = 0$.

$\mathrm{ran}(A_2)$ is not closed since A_2 is one-to-one, but not coercive.
(Set $\varphi_n = n^{-1/2}\, \chi_{[0,1/n]}$, then $||\varphi_n|| = 1$ but $||A_2 \varphi_n|| \to 0$.)

(d) $\sigma_{\rm p}(A_2) = \emptyset$, $\sigma_{\rm c}(A_2) = [-\pi/2,\, \pi/2]$, $\sigma_{\rm r}(A_2) = \emptyset$, and $||A_2|| = \pi/2$.

Motivation: Set $I = [-\pi/2,\, \pi/2]$. The facts that A_2 is self-adjoint and that $||A_2|| = \pi/2$ imply that $\sigma(A_2) \subseteq I$ and that $\sigma_{\rm r}(A_2) = \emptyset$. If $\lambda \in I$, then $A_2 - \lambda I$ is not onto (e.g. $e^{-x^2} \in L^2 \backslash ran(A_2 - \lambda I)$) so in fact $I = \sigma(A_2)$. Finally observe that $A_2 - \lambda I$ is one-to-one for any λ to see that $\sigma_{\rm p}(A_2) = \emptyset$.

Problem 3: (14p) Define for $x \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$ the functions

$$T_n(x) = \frac{n\,x}{n\,x^2 + 1}.$$

Does $(T_n)_{n=1}^{\infty}$ converge in $\mathcal{S}^*(\mathbb{R})$? If so, to what? Please motivate your answer.

Solution: First observe that for any non-zero x we have

$$\lim_{n\to\infty} T_n(x) = \frac{1}{x}$$

so it seems reasonable to guess that $T_n \to T$ where T denotes the principal value of $1/x$:

$$T(\varphi) = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}\,dx.$$

To prove that $T_n \to T$, we need to show that for any fixed $\varphi \in \mathcal{S}(\mathbb{R})$, we have $T_n(\varphi) \to T(\varphi)$.

Fix $\varphi \in \mathcal{S}(\mathbb{R})$ and set

$$\psi(x) = \frac{\varphi(x) - \varphi(-x)}{x}.$$

Now observe that

$$T_n(\varphi) = \int_{-\infty}^{\infty} \frac{nx}{nx^2+1}\varphi(x)\,dx = \int_0^{\infty} \frac{nx}{nx^2+1}(\varphi(x) - \varphi(-x))\,dx = \int_0^{\infty} \frac{nx^2}{nx^2+1}\psi(x)\,dx.$$

It follows that

$$|T_n(\varphi) - T(\varphi)| = \left|\int_0^{\infty}\left(\frac{nx^2}{nx^2+1} - 1\right)\,\psi(x)\,dx\right| \leq \int_0^{\infty} \frac{1}{nx^2+1}\,|\psi(x)|\,dx. \tag{1}$$

The integrand in (1) converges pointwise to zero, and it is bounded by the function $|\psi|$ which satisfies[1] $\int |\psi| < \infty$. The Lebesgue dominated convergence theorem then implies that

$$\lim_{n\to\infty} |T(\varphi) - T(\varphi_n)| = 0.$$

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[1]To explicitly prove that $\int|\psi| < \infty$, we observe that $\int_0^1 |\psi| \leq \int_0^1 2\,||\varphi||_{1,0}$ and that $\int_1^{\infty} |\psi| \leq \int_1^{\infty} \frac{1}{x^2}\,2\,||\varphi||_{0,1}$.

Problem 4: (22p) Consider the Banach space $X = L^5(\mathbb{R})$ equipped with the standard norm

$$||f||_5 = \left(\int_{\mathbb{R}} |f(x)|^5 \, dx \right)^{1/5}.$$

(a) (6p) What is X^*? Describe the action of an element of X^*.

(b) (6p) Which of the following statements are necessarily true (no motivation required):
 (i) Any bounded sequence in X has a weakly convergent subsequence.
 (ii) The weak-$\star$ topology on X is identical to the weak topology.
 (iii) Any bounded set $\Omega \subseteq X$ is pre-compact in the weak topology.
 (iv) Any bounded set $\Omega \subseteq X$ that is closed in the norm topology is compact in the weak topology.

(c) (10p) Let α be a real number and define the functions $(f_n)_{n=1}^{\infty}$ via

$$f_n(x) = n^{\alpha} \, \chi_{[n,\, n+1/n]}(x) = \begin{cases} 0 & x < n \\ n^{\alpha} & n \leq x \leq n + 1/n \\ 0 & n + 1/n < x. \end{cases}$$

For which α does $(f_n)_{n=1}^{\infty}$ converge in norm? Weakly? Motivate your answer carefully.

Solution:

(a) The dual of L^5 is L^q where $q = 1/(1 - 1/5) = 1/(4/5) = 5/4$. What this means is that for any functional $\varphi \in (L^5)^*$ there is a unique $g \in L^{5/4}$ such that

$$\varphi(f) = \int_{-\infty}^{\infty} f(x) \, g(x) \, dx.$$

(b) The true statements are (i), (ii), and (iii). (Since L^5 is reflexive, (ii) is true, and then Banach-Alaoglu implies (i) and (iii). (iv) is not true since Ω need not be compact in the weak topology. (E.g., $\Omega = \{f \in L^5 : ||f||_5 = 1\}$ is closed in the norm topology, but not in the weak topology.)

(c) First observe that $||f_n||_5^5 = \int_{-\infty}^{\infty} |f_n|^5 = n^{5\alpha} \int_n^{n+1/n} = n^{5\alpha} \frac{1}{n} = n^{5\alpha - 1}$.

Case 1 — $\alpha < 1/5$: In this case, $||f_n||_5 \to 0$ so (f_n) converges to zero in norm and weakly.

Case 2 — $\alpha > 1/5$: In this case, $||f_n||_5 \to \infty$ so (f_n) cannot converge either in norm or weakly.

Case 3 — $\alpha = 1/5$: In this case, $||f_n||_5 = 1$. Moreover, if $n \neq m$, we find $||f_n - f_m||_5^5 = ||f_n||_5^5 + ||f_m||_5^5 = 2$ so the sequence does not converge in norm. We claim that (f_n) converges weakly to zero. To prove this, pick $g \in L^{5/4} = (L^5)^*$. Then

$$\left| \int f_n \, g \right| \leq n^{1/5} \int_n^{n+1/n} |g| \leq \{\text{Hölder}\} \leq n^{1/5} \left(\int_n^{n+1/n} \right)^{1/5} \left(\int_n^{n+1/n} |g|^{5/4} \right)^{4/5} = \left(\int_n^{n+1/n} |g|^{5/4} \right)^{4/5}.$$

Now $\chi_{[n,n+1/n]} |g|^{5/4} \leq |g|^{5/4} \in L^1$, so by LDCT we find

$$\lim_{n\to\infty} \int_n^{n+1/n} |g|^{5/4} = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[n,n+1/n]} |g|^{5/4} = \{\text{LDCT}\} = \int_{\mathbb{R}} \left(\lim_{n\to\infty} \chi_{[n,n+1/n]} |g|^{5/4} \right) = 0.$$

It follows that $\lim_{n\to\infty} \left| \int f_n g \right| = 0$.

Problem 5: (24p) Let h and g be measurable functions on $\mathbb{R}$, and let $(h_n)_{n=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Suppose that $h_n\, g \in L^1(\mathbb{R})$ for all n, and that

$$\lim_{n\to\infty} h_n(x) = h(x) \qquad \text{for every } x \in \mathbb{R}.$$

Please answer the following questions, and provide brief motivations:

(a) (8p) Suppose that h_n and g are non-negative and that $\int h_n\, g = 1/n$.
Is it necessarily the case that $\int h\, g = 0$?

(b) (8p) Suppose that $|h_n(x)| \leq |h(x)|$ for all x and n, that $h \in L^2(\mathbb{R})$, and that $g \in L^2(\mathbb{R})$.
Is it necessarily the case that $\displaystyle\lim_{n\to\infty} \int h_n\, g = \int h\, g$?

(c) (8p) Suppose that $0 \leq h_1(x) \leq h_2(x) \leq h_3(x) \leq \cdots$ for all x and set $c_n = \int h_n\, g$.
Is the sequence $(c_n)_{n=1}^{\infty}$ necessarily convergent?
(And yes, if $c_n \to \infty$ or $c_n \to -\infty$, we do say that (c_n) is convergent.)

Solution: Set $f_n = h_n\, g$ and $f = h\, g$. Then f_n converges pointwise to f.

(a) Yes. Since f_n are non-negative, Fatou's lemma applies:

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

Now observe that

$$\liminf f_n = hg$$

and

$$\liminf \int f_n = \liminf (1/n) = 0.$$

(b) Yes, this follows from the Lebesgue dominated convergence theorem and Cauchy-Schwartz. (Observe that $|h_n g| \leq |f|$, and $\int |f| = \int |hg| \leq ||h||_{L^2}\, ||g||_{L^2} < \infty$.)

(c) No. All the convergence theorems are violated since the integrand $h_n g$ need not be non-negative, and no "dominator" need exist. For a counter-example, consider

$$h_n = \chi_{[0,\,n)} \qquad g = \sum_{n=0}^{\infty} (-1)^n \chi_{[n,n+1)}.$$

Then

$$c_n = \int_{\mathbb{R}} h_n g = \int_0^n g = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Grading guide:

(1) —

(2) In parts (a) and (c), the closed range part is worth 2 points and the other questions are worth 1. If motivations are overall good, that gives one additional points.

(3) Observe that the function $\frac{1}{x}\varphi(x)$ is not integrable in the Lebesgue sense, and you have to explicitly deal with the principal value.

Among solutions that did deal with the principal value, many were cavalier about interchanging the limits $\epsilon \to 0$ and $n \to \infty$.

(Disturbingly, many solutions incorrectly evaluated the limit of $\dfrac{nx}{nx^2+1}$ as $n \to \infty$.)

(4) In (4b), two points were deducted for each incorrect answer.

(5) 3p for each correct answer, and a max of 5p for each correct motivation.

In problem (c), note that when dealing with convergence of integrals, we say that a sequence of numbers that converges to infinity (or minus infinity) is "convergent." This is a result of working with the extended real numbers $\overline{\mathbb{R}}$. Observe that convergence to infinity is a permissible outcome of the "monotone *convergence* theorem" for instance. As a result, an example of functions h_n and g that satisfy the assumptions and such that $\int h_n g \to \infty$ is not really a counter-example. However, given that this misunderstanding was ubiquitous, it is possible that this point was not emphasized sufficiently in class, and such a "counter-example" earned 7 out of the maximal 8 points.