Applied Analysis (APPM 5450): Midterm 3
11.35am – 12.50pm, April 23, 2008. Closed books.

Problem 1: Mark the following as TRUE/FALSE. Motivate your answers briefly.

(a) [2p] If $f_n \rightharpoonup f$ in $L^2(\mathbb{R}^d)$, then $\hat{f}_n \rightharpoonup \hat{f}$ in $L^2(\mathbb{R}^d)$. (Note the *weak* convergence arrows.)

(b) [2p] Set $B = \{f \in L^2(\mathbb{R}^d) : \, ||f||_2 \leq 1\}$. Then $\mathcal{F}$ is a bijection from B to B.

(c) [2p] Let f be a function on $\mathbb{R}$ such that $\displaystyle\int_{-\infty}^{\infty} (1 + |x|)\, |f(x)|\, dx < \infty$. Then $\hat{f} \in C^1(\mathbb{R})$.

(d) [2p] If $f_n \to f$ in $L^1(\mathbb{R}^d)$, then $\hat{f}_n \to \hat{f}$ uniformly.

(e) [2p] If $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$ and α is a multi-index, then $\partial^\alpha \hat{\varphi}_n \to \partial^\alpha \hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^d)$.

Solution:

(a) TRUE.

Note that $f_n \rightharpoonup f \quad \Leftrightarrow \quad \langle f_n,\, g\rangle \to \langle f,\, g\rangle \ \forall g \in L^2$.

Since $\mathcal{F}$ preserves the inner product: $\langle f_n,\, g\rangle \to \langle f,\, g\rangle \ \forall g \in L^2 \quad \Leftrightarrow \quad \langle \hat{f}_n,\, \hat{g}\rangle \to \langle \hat{f},\, \hat{g}\rangle \ \forall g \in L^2$.

Since $\mathcal{F}$ is bijective: $\langle \hat{f}_n,\, \hat{g}\rangle \to \langle \hat{f},\, \hat{g}\rangle \ \forall g \in L^2 \quad \Leftrightarrow \quad \langle \hat{f}_n,\, g\rangle \to \langle \hat{f},\, g\rangle \ \forall g \in L^2$.

(b) TRUE.
$\mathcal{F}$ is an isometry.

(c) TRUE.
Note that $\hat{f}' = \mathcal{F}[-i\, x\, f(x)]$.
Since $x\, f(x) \in L^1$, the Riemann-Lebesgue lemma then asserts that $\hat{f}' \in C_0(\mathbb{R})$.

(d) TRUE.
Note that $||\hat{f} - \hat{f}_n||_{\text{u}} = \sup_t \left|\beta^d \int e^{-i\,x\,t}\big(f(x) - f_n(x)\big)\, dx\right| \leq \beta^d \int |f - f_n| = \beta^d ||f - f_n||_{L^1}$.

(e) TRUE.
Both $\mathcal{F}$ and ∂^α are continuous maps on $\mathcal{S}(\mathbb{R}^d)$. Then their composition must by continuous too.

Problem 2: [7p] Let d be a positive integer. Prove that if s is a real number that is "large enough", then $H^s(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$. Make sure to specify what "large enough" is.

Solution: The Riemann-Lebesgue lemma says that if $f \in L^1$, then $\hat{f} \in C_0$. It follows that if $\hat{f} \in L^1$, then $\hat{\hat{f}} \in C_0$. Noting that $\hat{\hat{f}}(x) = f(-x)$, we see that $\hat{f} \in L^1 \Rightarrow f \in C_0$.

We will prove that if $f \in H^s$ for sufficiently large s, then $\hat{f} \in L^1$.

Suppose that $f \in H^s$. Then

$$\int |\hat{f}(t)|\,dt = \int (1+|t|^2)^{-s/2}\,(1+|t|^2)^{s/2}\,|\hat{f}(t)|\,dt \leq \{\text{Cauchy-Schwartz}\}$$
$$\leq \left(\int (1+|t|^2)^{-s}\,dt\right)^{1/2} \left(\int (1+|t|^2)^{s}\,|\hat{f}(t)|^2\,dt\right)^{1/2} = \left(\int (1+|t|^2)^{-s}\,dt\right)^{1/2} ||f||_{H^s}.$$

Noting that $\int_{\mathbb{R}^d} (1+|t|^2)^{-s}\,dt < \infty$ if $-2s < -d$, we find that $\hat{f} \in L^1$ if $\boxed{s > d/2.}$

Problem 3: Calculate the Fourier transform of the following functions on $\mathbb{R}$:

(a) [3p] The Dirac δ-function.

(b) [3p] $f(x) = x^k$.

(c) [3p] $g(x) = \sin(x)$.

Solution:

(a) Fix a $\varphi \in \mathcal{S}$. Then

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}} \int e^{-i\,0\,x}\, \varphi(x)\, dx = \frac{1}{\sqrt{2\pi}} \int \varphi(x)\, dx = \langle \frac{1}{\sqrt{2\pi}}, \varphi \rangle.$$

It follows that $\hat{\delta} = 1/\sqrt{2\pi}$.

(b) Note that $\mathcal{F}[(-i\,x)^k\, T] = \partial^k\, \hat{T}$ for any function T. Set $T = 1$. Then from part (a) we find that $\hat{T} = \sqrt{2\pi}\,\delta$ and so $\mathcal{F}[x^k] = \mathcal{F}[i^k\,(-i\,x)^k\, T] = i^k\,\partial^k\,\hat{T} = i^k\,\sqrt{2\,\pi}\,\partial^k\,\delta$.

(c) Note that $\mathcal{F}[e^{i\,b\,x}\, T] = \hat{T}(x-b)$. It follows that, again with $T = 1$,

$$\mathcal{F}[\sin(x)] = \mathcal{F}[\frac{1}{2i}\left(e^{i\,x} - e^{-i\,x}\right) T] = \frac{1}{2i}\left(\hat{T}(x-1) - \hat{T}(x+1)\right) = \frac{\sqrt{2\,\pi}}{2i}\left(\delta(x-1) - \delta(x+1)\right).$$

Problem 4:

(a) [2p] State the definition of a σ-algebra.

(b) [2p] Is every topology is a σ-algebra? Motivate your answer.

(c*) [2p] Is every σ-algebra a topology? Motivate your answer.

(d) [2p] State the definition of a *measure*.

(e) [4p] Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{\Omega_\beta\}_{\beta\in B}$ be a countable collection of sets in $\mathcal{A}$. Prove directly from the definition of a measure that

$$\mu\left(\bigcup_{\beta\in B}\Omega_\beta\right) = \sup\left\{\mu\left(\bigcup_{\beta\in C}\Omega_\beta\right)\ :\ C \text{ is a finite subset of } B\right\}. \tag{1}$$

Hint: Since B is countable, you may assume that $B = \{1, 2, 3, \dots\}$. Then the statement you are asked to prove is equivalent to the statement $\mu\left(\bigcup_{n=1}^{\infty}\Omega_n\right) = \sup\left\{\mu\left(\bigcup_{n=1}^{N}\Omega_n\right)\ :\ N = 1, 2, 3, \dots\right\}$.

(f*) [2p] Demonstrate that the formula (1) is not necessarily true if B is uncountable.

Solution:

(a) See the text book.

(b) No. Consider for instance the standard topology on $\mathbb{R}$. The open interval $(-1, 1)$ belongs to the standard topology, but its complement (which is a closed set) does not.

(c) No. As a counter example, set $X = \mathbb{R}$, and let $\mathcal{A}$ denote the collection of subsets of X that are either countable, or have countable complements. This set is closed under countable unions and intersections and is consequently a σ-algebra. But it is not a topology since it is not closed under arbitrary unions. (For instance, for any $x \in \mathbb{R}$, the set $\Omega_x = \{x\} \in \mathcal{A}$, but $\bigcup_{x\geq 0}\Omega_x = [0, \infty) \notin \mathcal{A}$.)

(d) Set $A_1 = \Omega_1$, and define for $n = 2, 3, 4, \dots$ the sets A_n via

$$A_n = \Omega_n \setminus \left(\bigcup_{j=1}^{n-1}\Omega_j\right).$$

Then

$$\bigcup_{n=1}^{\infty}\Omega_n = \bigcup_{n=1}^{\infty}A_n, \qquad \text{and} \qquad \bigcup_{n=1}^{N}\Omega_n = \bigcup_{n=1}^{N}A_n,$$

but the unions of A_n's are unions of disjoint sets. It follows from the definition of a measure that

$$\mu\left(\bigcup_{n=1}^{\infty}\Omega_n\right) = \mu\left(\bigcup_{n=1}^{\infty}A_n\right) = \sum_{n=1}^{\infty}\mu(A_n) = \sup_N \sum_{n=1}^{N}\mu(A_n) = \sup_N \mu\left(\bigcup_{n=1}^{N}A_n\right) = \sup_N \mu\left(\bigcup_{n=1}^{N}\Omega_n\right).$$

(f) Consider the standard Lebesgue measure on $\mathbb{R}$. Define for $x \in \mathbb{R}$ the sets $\Omega_x = \{x\}$. Then $\mu\left(\cup_{x\in\mathbb{R}}\Omega_x\right) = \mu(\mathbb{R}) = \infty$, but $\mu\left(\cup_{x\in C}\Omega_x\right) = \mu(C) = 0$ for any finite subset C of $\mathbb{R}$.

Problem 5: [6p] We define for $n = 1, 2, 3, \dots$ functions f_n on $\mathbb{R}$ by $f_n(x) = n^{3/2}\, x\, e^{-n\,x^2}$. Either prove that $(f_n)_{n=1}^{\infty}$ does not converges in $\mathcal{S}^*(\mathbb{R})$, or give the limit point and prove convergence.

Solution: Set $g_n(x) = -\frac{1}{2}\sqrt{n}\, e^{-n\,x^2}$. Then $f_n = g_n'$. Define the finite real number α via

$$\alpha = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{2}$$

Then

$$\int_{-\infty}^{\infty} g_n(x)\, dx = \int_{-\infty}^{\infty} -\frac{1}{2}\sqrt{n}\, e^{-n\,x^2}\, dx = \{x = y/\sqrt{n}\} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2}\, dy = \alpha.$$

Since all g_n's are strictly negative, and their mass concentrate to the origin, it follows that $g_n \to \alpha\,\delta$ in $\mathcal{S}^*$ (here we use the fact that an approximate identity converges to δ in $\mathcal{S}^*$). Since differentiation is a continuous operation on $\mathcal{S}^*$ it follows that $g_n' \to \alpha\,\delta'$ and so $\boxed{f_n \to \alpha\,\delta'.}$

Remark: As it happens, $\alpha = -\frac{1}{2}\sqrt{\pi}$, but giving this constant implicitly as in (2) gives full credit for the problem.