Applied Analysis (APPM 5450): Midterm 1
11.35am – 12.50pm, Feb. 18, 2008. Closed books.

Problem 1: Let H be a Hilbert space with an ON-basis $(\varphi_j)_{j=1}^{\infty}$, and let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, $(u_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, and $(w_n)_{n=1}^{\infty}$ be sequences in H for which you know the following:

$\langle x_n,\, x_m\rangle = 0$ if $m \neq n$ and $\langle x_n,\, x_n\rangle = 1$.

$||y_n|| = 1$

$\limsup_{n\to\infty} ||z_n|| = \infty$

$||u_n|| = 1/n$ and $\lim_{n\to\infty} \langle\varphi_j,\, u_n\rangle = 0$ for every j.

$\lim_{n\to\infty} \langle\varphi_j,\, v_n\rangle = 0$ for every j.

There exists a $w \in H$ such that $||w_n|| \to ||w||$ and $\lim_{n\to\infty} \langle\varphi_j,\, w_n\rangle = \langle\varphi_j,\, w\rangle$ for every j.

Solution:

	x_n	y_n	z_n	u_n	v_n	w_n
(1) Necessarily converges strongly. (2) Does not converge strongly. (3) May or may not converge strongly.	2	3	2	1	3	1
(1) Necessarily has a strongly convergent subsequence. (2) Does not have a strongly convergent subsequence. (3) May or may not have a strongly convergent subsequence.	2	3	3	1	3	1
(1) Necessarily converges weakly. (2) Does not converge weakly. (3) May or may not converge weakly.	1	3	2	1	3	1
(1) Necessarily has a weakly convergent subsequence. (2) Does not have a weakly convergent subsequence. (3) May or may not have a weakly convergent subsequence.	1	1	3	1	3	1

Some (non-required!) comments:

(x_n) is an ON-sequence. It converges weakly to zero but does not converge strongly.

(y_n) necessarily has a weakly convergent subsequence since the unit ball in a Hilbert space is weakly compact.

(z_n) itself cannot converge either weakly or strongly since it has a subsequence (z_{n_j}) such that $\lim_{j\to\infty} ||z_{n_j}|| = \infty$. However, it may have convergent sequences interlaced.

The condition $||u_n|| \to 0$ by itself implies that $u_n \to 0$ strongly (and hence weakly as well).

You cannot say anything. Both the sequence $v_n = n\,\varphi_n$ (which does not have any convergent subsequences) and the sequence $v_n = 0$ satisfy the given condition.

(w_n) is weakly convergent since it is a bounded sequence that converges "componentwise". Moreover, it must be that $w_n \rightharpoonup w$, and since in addition $||w_n|| \to ||w||$ strong convergence follows.

Problem 2: Set $I = [-\pi/2, \pi]$ and consider the Hilbert space $H = L^2(I)$.

(a) Set $\varphi_n(x) = \sin(nx)$ and prove that the set $\mathcal{P} = \text{span}(\varphi_n)_{n=1}^\infty$ is not dense in H. (3p)

(b) Set $e_n(x) = e^{i\,n\,x}/\sqrt{2\pi}$ and prove that the set $(e_n)_{n=-\infty}^\infty$ is linearly dependent in the sense that there exists a sequence of complex numbers $(\alpha_n)_{n=-\infty}^\infty$ such that

$$0 < \sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty \qquad \text{and} \qquad \lim_{N\to\infty} ||\sum_{n=-N}^{N} \alpha_n\, e_n||_{L^2(I)} = 0.$$

(4p)

(c) Provide an ON-basis for H. (3p)

Solution:

(a) Set $\psi = \chi_{[-1/2,1/2]}$. Then $\langle \varphi_n,\, \psi\rangle = \int_{-1/2}^{1/2} \sin(n\,x)\,dx = 0$ so $\psi \in \mathcal{P}^\perp$. Consequently, for any $\varphi \in \mathcal{P}$, we have $||\psi - \varphi|| = \sqrt{||\psi||^2 + ||\varphi||^2} \geq ||\psi|| = 1$.

(b) Set $\psi = \chi_{[-\pi,-\pi/2]}$, and set

$$\alpha_n = \langle e_n,\, \psi\rangle_{L^2([-\pi,\pi])} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{-\pi/2} e^{-i\,n\,x}\,dx.$$

Since (e_n) is an ON-basis for $L^2([-\pi,\pi])$, we have $\sum |\alpha_n|^2 = ||\psi||^2_{L^2([-\pi,\pi])} = \pi/2$.

Next, set $\psi_N = \sum_{n=-N}^{N} \alpha_n\, e_n$. Then

$$||\sum_{n=-N}^{N} \alpha_n\, e_n||^2_{L^2(I)} = \int_{-\pi/2}^{\pi} |\psi_N(x)|^2\,dx = \int_{-\pi/2}^{\pi} |\psi_N(x) - \psi(x)|^2\,dx$$
$$\leq \int_{-\pi}^{\pi} |\psi_N(x) - \psi(x)|^2\,dx = ||\psi_N - \psi||^2_{L^2([-\pi,\pi])} \to 0.$$

(c) There are many choices. For instance:

$$\big\{A_n\, \sin(n\,(2/3)\,(x+\pi/2))\big\}_{n=1}^\infty, \qquad \text{where} \qquad A_n = \frac{1}{||\sin(n\,(2/3)\,(x+\pi/2))||_{L^2(I)}}$$

$$\big\{B_n\, e^{i\,n\,(4/3)\,(x-\pi/4)}\big\}_{n=-\infty}^\infty, \qquad \text{where} \qquad B_n = \frac{1}{||e^{i\,n\,(4/3)\,(x-\pi/4)}||_{L^2(I)}} = \sqrt{\frac{2}{3\,\pi}}$$

$$\big\{C_n\, e^{i\,n\,(4/3)\,x}\big\}_{n=-\infty}^\infty, \qquad \text{where} \qquad C_n = \frac{1}{||e^{i\,n\,(4/3)\,x}||_{L^2(I)}} = \sqrt{\frac{2}{3\,\pi}}$$

Problem 3: Let $(\lambda_n)_{n=-\infty}^{\infty}$ denote a bounded sequence of complex numbers and consider the map

$$A:\ L^2(\mathbb{T}) \to l^2(\mathbb{Z}):\ u \mapsto v = (\dots,\, v_{-1},\, v_0,\, v_1,\, \dots) \text{ where } v_n = \lambda_n\,\langle e_n,\, u\rangle. \tag{1}$$

In (1), e_n denotes the Fourier basis for $L^2(\mathbb{T})$, $e_n(x) = e^{i\,n\,x}/\sqrt{2\,\pi}$.

(a) Prove that $||A|| = \sup_n |\lambda_n|$. (4p)

(b) Let $\mathcal{F}:\ L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ denote the Fourier transform. Complete the following sentences:

$\mathcal{F}^{-1}A$ is *self-adjoint* if and only if every number λ_n satisfies ...

$\mathcal{F}^{-1}A$ is *unitary* if and only if every number λ_n satisfies ...

Motivate briefly. (6p)

Solution:

(a) Set $M = \sup|\lambda_n|$. Then

$$||A\,u||^2 = \sum_n |\lambda_n\,\langle e_n,\, u\rangle|^2 \leq \sum_n M^2\,|\,\langle e_n,\, u\rangle|^2 = ||u||^2,$$

so $||A|| \leq M$. Conversely,

$$||A|| = \sup_{||u||=1} ||A\,u|| \geq \sup_n ||A\,e_n|| = \sup_n |\lambda_n| = M.$$

(b) We have

$$[\mathcal{F}^{-1}\,A]\,u = \sum_n \lambda_n\,\langle u,\, e_n\rangle\,e_n.$$

It is clear that

$$[(\mathcal{F}^{-1}\,A)^*]\,u = \sum_n \bar{\lambda}_n\,\langle u,\, e_n\rangle\,e_n,$$

so $\mathcal{F}^{-1}\,A$ is self-adjoint iff $\bar{\lambda}_n = \lambda_n$ (which is to say, iff λ_n is real for all n).

Similarly: $\mathcal{F}^{-1}\,A$ is unitary iff $(\mathcal{F}^{-1}\,A)^* = (\mathcal{F}^{-1}\,A)^{-1}$. It follows that

$$\mathcal{F}^{-1}\,A \text{ is unitary} \quad \Leftrightarrow \quad \bar{\lambda}_n = \lambda_n^{-1}\ \forall n \quad \Leftrightarrow \quad |\lambda_n| = 1\ \forall n.$$

Problem 4: Recall that for an $n \times n$ matrix A it is the case that

$$\operatorname{ran}(A) = \ker(A^*)^\perp. \tag{2}$$

Now consider the Hilbert space $H = L^2([-\pi,\, \pi])$ and the operator

$$[A\,u](x) = x\, e^{i\,x}\, u(x).$$

(a) Construct A^* and prove that (2) does not hold for A. (6p)

(b) Determine $||A||$. (4p)

Solution:

(a) First we note that

$$\langle A\,u,\, v\rangle = \int \overline{x\, e^{i\,x}\, u(x)}\, v(x)\, dx = \int \overline{u(x)}\, (x\, e^{-i\,x}\, v(x))\, dx = \langle u,\, A^*\, v\rangle,$$

with

$$[A^* v](x) = x\, e^{-i\,x}\, v(x).$$

It is immediately clear that $\ker(A^*) = \{0\}$ so $\ker(A^*)^\perp = H$.

However, A is not onto. To see this, note that the constant function 1 belongs to H, but the equation $A\,u = 1$ does not have a solution in H. It follows that (2) does not hold.

(b) First we note that

$$||A\,u||^2 = \int |x, e^{i\,x}\, u(x)|^2\, dx \leq \int \pi^2 |u(x)|^2\, dx = \pi^2 ||u||^2.$$

It follows that $||A|| \leq \pi$. To prove the converse, consider the functions

$$u_n = \sqrt{n}\, \chi_{[\pi - 1/n,\, \pi]}.$$

We have $||u_n|| = 1$ and so

$$\begin{aligned} ||A||^2 = \sup_{||u||=1} ||A\,u||^2 \geq \sup_n ||A\,u_n||^2 &= \sup_n \int_{\pi-1/n}^{\pi} |x\, e^{i\,x}\, \sqrt{n}|^2\, dx \\ &\geq \sup_n \int_{\pi-1/n}^{\pi} (\pi - 1/n)^2\, n\, dx = \sup_n\, (\pi - 1/n)^2 = \pi^2. \end{aligned}$$

Problem 5: Let H_1 and H_2 be Hilbert spaces.

(a) Define what it means for a map $U \in \mathcal{B}(H_1,\, H_2)$ to be *unitary.* (2p)

(b) Suppose that $A \in \mathcal{B}(H_1,\, H_2)$, that A is onto, and that $||A\,u|| = ||u||$ for all $u \in H_1$. Is A necessarily unitary? Motivate briefly. (2p)

Solution:

(a) A map $U \in \mathcal{B}(H_1, H_2)$ is *unitary* if it is bijective and

$$\langle U\,x,\, U\,y\rangle_{H_2} = \langle x,\, y\rangle_{H_1} \qquad \forall\, x, y \in H_1.$$

(b) Suppose that A is an isometry and is onto. Since A is an isometry, it follows that A is one-to-one, and hence bijective. To see that A preserves the inner product, simply use a spectral identity:

$$\langle U\,x,\, U\,y\rangle_{H_2} = \frac{1}{4}\big(||U\,x + U\,y||^2_{H_2} - ||U\,x - U\,y||^2_{H_2} - i\,||U\,x + i\,U\,y||^2_{H_2} + i\,||U\,x - i\,U\,y||^2_{H_2}\big)$$
$$\frac{1}{4}\big(||x + y||^2_{H_1} - ||x - y||^2_{H_1} - i\,||x + i\,y||^2_{H_1} + i\,||x - i\,y||^2_{H_1}\big) = \langle x,\, y\rangle_{H_1}$$