Homework set 3 — APPM5450, Spring 2008

From the textbook: 8.6. Optional: 8.5.

Problem 1: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for H. Given a bounded sequence of complex number $(\lambda_n)_{n=1}^{\infty}$, define the operator A by setting $A\,u = \sum_{n=1}^{\infty} \lambda_n\,\varphi_n\,\langle\varphi_n,\,u\rangle$.

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Prove that $A^{*}u = \sum_{n=1}^{\infty} \bar{\lambda}_n\varphi_n\,\langle\varphi_n,\,u\rangle$. Conclude that A is self-adjoint iff all λ_n's are real. When is A skew-symmetric?

Problem 2: Consider the Hilbert space $H = L^2([-\pi,\,\pi])$, and the operator $A \in \mathcal{B}(H)$ defined by $[A\,u](x) = |x|\,u(x)$. Prove that A is self-adjoint and positive, but not coercive. Prove that

$$\langle u,\,v\rangle_A = \langle A\,u,\,v\rangle$$

is an inner product on H, but that the topology generated by (the norm generated by) this inner product is *not* equivalent to the topology generated by the L^2-norm.

Problem 3: Set $H = l^2(\mathbb{Z})$ and let R denote the right-shift operator (so that if $y = R\,x$, then $y_n = x_{n-1}$). Construct R^{*}. Prove that R is unitary. (Recall that the right-shift operator on $l^2(\mathbb{N})$ is *not* unitary!)

Problem 4: Consider the Hilbert space $L^2(\mathbb{T})$. Let k denote a continuous function on $\mathbb{T}^2$ that takes on complex values. Let A denote the operator $[Au](x) = \int_{\mathbb{T}} k(x,y)\,u(y)\,dy$. Prove that $[A^{*}u](x) = \int_{\mathbb{T}} \overline{k(y,x)}\,u(y)\,dy$. Conclude that A is self-adjoint iff $k(x,y) = \overline{k(y,x)}\ \forall\ x,y \in \mathbb{T}$.

Problem 5 (optional): Assume that you have proved that in a Hilbert space, $\text{ran}(I-K)$ is closed whenever K has finite rank. Prove from this that $\text{ran}(I-K)$ is closed for all compact operators K. (Then if you feel industrious, feel free to try proving the statement given in the first sentence.)