Homework set 12 — APPM5450, Spring 2007 — Solutions

Problem 11.22: Set $T = \text{sign}(t)$. We seek to prove that $\check{T} = \alpha\text{PV}(1/x)$ for some α.

For $N = 1, 2, 3, \dots$, set $T_N = \chi_{[-N,N]}\,T$. Then $T_N \to T$ in $\mathcal{S}^*$ since for any $\varphi \in \mathcal{S}$, we have

$$\langle T_n,\, \varphi\rangle = \int_{-N}^{N} \text{sign}(x)\,\varphi(x)\,dx \to \int_{-\infty}^{\infty} \text{sign}(x)\,\varphi(x)\,dx = \langle T,\, \varphi\rangle.$$

Since the Fourier transform is a continuous operator on $\mathcal{S}^*$, we know that $\check{T}$ is the limit of the sequence $(\check{T}_N)_{N=1}^{\infty}$.

Since $T_N \in L^1$, we can compute $\check{T}_N$ by directly evaluating the integral. We find that

$$\check{T}_N(x) = \beta\frac{1-\cos(N\,x)}{x} \tag{1}$$

for some constant β. If $\varphi \in \mathcal{S}$, then

$$\begin{aligned}\langle\frac{1-\cos(N\,x)}{x},\, \varphi\rangle &= \int_{\mathbb{R}} \frac{1-\cos(N\,x)}{x}\,\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{1-\cos(N\,x)}{x}\,\varphi(x)\,dx \\ &= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{1}{x}\varphi(x)\,dx - \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \cos(N\,x)\,\frac{1}{x}\varphi(x)\,dx \\ &= \langle\text{PV}(1/x),\, \varphi\rangle - \langle\cos(N\,x)\,\text{PV}(1/x),\, \varphi\rangle.\end{aligned}$$

It follows that formula (1) can be written $\check{T}_N(x) = \beta\,\text{PV}(1/x) - \beta\,\cos(N\,x)\,\text{PV}(1/x)$.

It remains to prove that $\cos(N\,x)\,\text{PV}(1/x) \to 0$ in $\mathcal{S}'$. We find that

$$\begin{aligned}\langle\cos(N\,x)\,\text{PV}(1/x),\, \varphi\rangle &= \langle\text{PV}(1/x),\, \cos(N\,x)\,\varphi\rangle \\ &= \int_0^{\infty} \cos(N\,x)\,\frac{1}{x}\,\varphi(x)\,dx + \int_{-\infty}^{0} \cos(N\,x)\,\frac{1}{x}\varphi(x)\,dx \\ &= \int_0^{\infty} \cos(N\,x)\,\frac{\varphi(x)-\varphi(-x)}{x}\,dx.\end{aligned}$$

Now set $\psi(x) = \frac{\varphi(x)-\varphi(-x)}{x}$. Then ψ is a continuously differentiable, quickly decaying function on $[0,\,\infty)$, so we can perform a partial integration to obtain

$$\begin{aligned}\left|\int_0^{\infty} \cos(N\,x)\,\frac{\varphi(x)-\varphi(-x)}{x}\,dx\right| &= \left|\left[\frac{\sin(N\,x)}{N}\,\psi(x)\right]_0^{\infty} - \int_0^{\infty}\frac{\sin(N\,x)}{N}\,\psi'(x)\,dx\right| \\ &\leq \frac{1}{N}\int_0^{\infty} |\psi'(x)|\,dx.\end{aligned}$$

If we can prove that $\int_0^{\infty} |\psi'(x)|\,dx < \infty$, we will be done. First note that for $x \in [0,\,1]$, $\psi(x) = 2\varphi'(0) + O(x^2)$, so for $x \in [0,\,1]$, we have $|\psi'(x)| \leq C_1$ for some finite C_1. For $x \in [1, \infty)$, we have

$$|\psi'(x)| = \left|\frac{\varphi'(x)+\varphi'(-x)}{x} - \frac{\varphi(x)-\varphi(-x)}{x^2}\right| \leq 2\frac{||\varphi||_{1,1}}{x^2} + 2\frac{||\varphi||_{0,0}}{x^2} = \frac{C_2}{x^2}.$$

and so

$$\int_0^{\infty} |\psi'(x)|\,dx \leq \int_0^1 C_1\,dx + \int_1^{\infty}\frac{C_2}{x^2}\,dx < \infty.$$

Problem 12.2: We use "$\Cup$" to denote disjoint unions.

(a) Suppose that $A, B \in \mathcal{A}$. Then note that $A\backslash B = A \cap B^{c} = (A^{c} \cup B)^{c}$. It now follows directly from the axioms that $A\backslash B \in \mathcal{A}$.

(b) Set $C = B\backslash A$. Then $B = A \Cup C$, so

$$\mu(B) = \mu(A \Cup C) = \mu(A) + \mu(C) \geq \mu(A).$$

(c) Set $C = A \cap B$. Then $A = (A\backslash B) \Cup C$ and $B = (B\backslash A) \Cup C$ so

$$\begin{aligned}\mu(A \cup B) = \mu((A\backslash B) \Cup C \Cup (B\backslash A)) &= \mu(A\backslash B) + \mu(C) + \mu(B\backslash A) \\ &\leq \mu(A\backslash B) + \mu(C) + \mu(C) + \mu(B\backslash A) \\ &= \mu((A\backslash B) \Cup C) + \mu(C \Cup (B\backslash A)) = \mu(A) + \mu(B).\end{aligned}$$

Problem 12.3: The trick is to write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union. For $n = 1, 2, 3, \ldots$ set $B_n = A_{n+1}\backslash A_n$. Then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n \right),$$

where there union on the right is a disjoint one. Now use additivity twice:

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N\to\infty}\left(\mu(A_1) + \sum_{n=1}^{N} \mu(B_n)\right) = \lim_{N\to\infty} \mu\left(A_1 \cup \left(\bigcup_{n=1}^{N} B_n\right)\right) = \lim_{N\to\infty} \mu\left(A_N\right).\end{aligned}$$

For the second part, set $C = \cap_{n=1}^{\infty} A_n$ and $C_n = A_n\backslash A_{n+1}$. Then

$$\mu(A_N) = \mu\left(C \cup \left(\bigcup_{n=N}^{\infty} C_n\right)\right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).$$

Since $\infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n)$, we find that

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,$$

which completes the proof. For the counterexample, consider $X = \mathbb{R}^2$, and $A_n = \{x = (x_1, x_2) : |x_2| < 1/n\}$. Then $\mu(A_n) = \infty$ for all n, but $\cap_{n=1}^{\infty} A_n$ is the x_1-axis, which has measure zero.

Problem 12.5: Straight-forward.

Problem 12.7:

Reflexivity: It is obvious that $f(x) = f(x)$ a.e.

Symmetry: If $f(x) = g(x)$ a.e., then obviously $g(x) = f(x)$ a.e.

Transitivity: Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

$$A = \{x : f(x) \neq g(x)\}$$
$$B = \{x : g(x) \neq h(x)\}$$
$$C = \{x : f(x) \neq h(x)\}.$$

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0$.