Applied Analysis (APPM 5450): Midterm 3 – Solutions
5.00pm – 6.20pm, Apr 24, 2006. Closed books.

Note: In your solutions, explicitly state if you use an integral sign that does **not** refer to a Lebesgue integral. (All integrals in the exam are Lebesgue integrals.)

Problem 1: In this question, $(X, \mathcal{A}, \mu)$ denotes a measure space.

(a) What axioms must $\mathcal{A}$ satisfy? (2p)

(b) What axioms must μ satisfy? (2p)

(c) Prove that if Ω_1, $\Omega_2 \in \mathcal{A}$, then $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Given an exact condition for when equality occurs. ("Equality occurs if and only if ...") (2p)

(d) Define the Lebesgue integral of a measurable non-negative function. (2p)

(e) Define the "essential supremum" of a measurable function f on X. (2p)

(f) For which extended real numbers p are the simple functions dense in $L^p(\mathbb{R}^d)$? When is $C_c^\infty(\mathbb{R}^d)$ dense in L^p? (2p)

(a,b) See textbook.

(c) Set $A = \Omega_1 \backslash \Omega_2$, $B = \Omega_2 \backslash \Omega_1$, and $C = \Omega_1 \cap \Omega_2$. Then A, B, and C are all disjoint, and $\Omega_1 \cup \Omega_2 = A \cup B \cup C$. By the additivity of μ, we get

$$\mu(\Omega_1 \cup \Omega_2) = \mu(A \cup B \cup C) = \mu(A) + \mu(B) + \mu(C).$$

Moreover, $\Omega_1 = A \cup C$ and $\Omega_2 = B \cup C$ so

$$\mu(\Omega_1) + \mu(\Omega_2) = \mu(A \cup C) + \mu(B \cup C) = \mu(A) + \mu(B) + 2\mu(C).$$

Since $\mu(C) \geq 0$, it follows that $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$. Equality holds if and only if $\mu(C) = 0$; in other words, if $\Omega_1 \cap \Omega_2$ is a null-set.

(d) For a simple function $\varphi = \sum_{j=1}^n c_j \, \chi_{\Omega_j}$, we define $\int \varphi = \sum_{j=1}^n c_j \, \mu(\Omega_j)$. For a general non-negative f, we define $\int f = \sup\{\int \varphi : \ \varphi \text{ is simple, and } \varphi \leq f\}$.

(e) $\operatorname*{ess\,sup}_{x \in X} f(x) = \inf\{M : \ \mu(\{x : \ f(x) \geq M\}) = 0\} = \inf\{M : \ f(x) \leq M \text{ a.e.}\}$.

(f) The simple functions are dense in $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.
$C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

Problem 2: Define the function f on $\mathbb{R}^2$ by $f(x_1, x_2) = x_1\,x_2^2$. Define a tempered distribution T_f by $\langle T_f,\,\varphi\rangle = \int_{\mathbb{R}^2} f(x)\,\varphi(x)\,dx$. What is the Fourier transform of T_f? Motivate your answer carefully. (4p)

Let U denote the tempered distribution $\langle U,\,\varphi\rangle = \int \varphi$. Recall that $\hat{U} = 2\pi\delta$ (since $\langle \hat{U},\,\varphi\rangle = \langle U,\,\hat{\varphi}\rangle = \int \hat{\varphi}$, and by the Fourier inversion formula $\int \hat{\varphi} = 2\pi\varphi(0)$).

Note that $T_f = x_1\,x_2^2\,U = -i(-ix)^{(1,2)}\,U$.

Using the formula $\partial^\alpha\,\hat{T} = \mathcal{F}[(-ix)^\alpha\,T]$ with $\alpha = (1,2)$, we find that

$$\mathcal{F}[T_f] = \mathcal{F}[-i(-ix)^\alpha U] = -i\partial^\alpha \hat{U} = -i\,2\,\pi\,\frac{\partial^3}{(\partial x_1)(\partial x_2)^2}\delta.$$

In other words, $\langle \hat{T}_f,\,\varphi\rangle = -i\,2\,\pi\,\dfrac{\partial^3\varphi}{(\partial x_1)(\partial x_2)^2}(0).$

Problem 3: Calculate the limit

$$\lim_{n\to\infty} \int_n^{n+1} \sqrt{x}\, \tan(1/\sqrt{x})\, dx.$$

Motivate your answer carefully. (4p)

Changing variables to $y = x - n$, we have

$$I_n = \int_n^{n+1} \sqrt{x}\ \tan\frac{1}{\sqrt{x}}\, dx = \int_0^1 \sqrt{y+n}\ \tan\frac{1}{\sqrt{y+n}}\, dy.$$

Set $f_n(y) = \sqrt{y+n}\ \tan\frac{1}{\sqrt{y+n}}$. Then, for every $y \in [0,\, 1]$, we have $\lim_{n\to\infty} f_n(y) = 1$.

Since $\tan t \le 2t$ when $t \in [0,\, 1]$, we have $|f_n(y)| \le 2$ when $n \ge 1$, and $y \in [0,\, 1]$.

Since $|f_n|$ is bounded by the integrable function $2\chi_{[0,\,1]}$, the Lebesgue dominated convergence theorem applies, and we obtain

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^1 f_n(y)\, dy = \int_0^1 \lim_{n\to\infty} f_n(y)\, dy = \int_0^1 1\, dy = 1.$$

Remark: Lebesgue dominated convergence is overkill. It is trivial to show that $f_n \to 1$ in $C([0,\, 1])$ (*i.e.* uniformly). It then follows immediately that

$$\lim_{n\to\infty} \int f_n = \int 1 = 1.$$

Problem 4: Recall that for $s \in [0, \infty)$, the Sobolev space $H^s(\mathbb{R}^d)$ is defined as the set of all functions $f \in L^2(\mathbb{R}^d)$ such that $(1 + |t|^2)^{s/2}\, \hat{f}(t) \in L^2$. Prove that if s is large enough, then $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$. (4p)

The Riemann-Lebesgue lemma says that if $g \in L^1(\mathbb{R}^d)$, then $\hat{g} \in C_0(\mathbb{R}^d)$. It follows trivially that in order to prove that $f \in C_0$, it is sufficient to prove that $\hat{f} \in L^1$.

Assume that $f \in H^s$ for some $s \in [0, \infty)$. We have

$$||\hat{f}||_{L^1} = \int |\hat{f}| = \int |\hat{f}|\, (1 + |t|^2)^{s/2}\, \frac{1}{(1 + |t|^2)^{s/2}}.$$

Using the Cauchy-Schwartz inequality, we then find that

$$||\hat{f}||_{L^1} \leq \left(\int |\hat{f}|^2\, (1 + |t|^2)^s \right)^{1/2} \left(\int \frac{1}{(1 + |t|^2)^s} \right)^{1/2} = ||f||_{H^s} \left(\int \frac{1}{(1 + |t|^2)^s} \right)^{1/2}.$$

Now simply note that

$$\int \frac{1}{(1 + |t|^2)^s} = [\text{Area of unit sphere in } \mathbb{R}^d] \times \int_0^\infty \frac{1}{(1 + r^2)^s}\, r^{d-1}\, dr,$$

which is finite if $-2s + d - 1 < -1$, or, in other words, if $s > d/2$.