Applied Analysis (APPM 5450): Midterm 2
5.00pm – 6.20pm, Mar 22, 2006. Closed books.

Note: The problems are worth two points each, for a total of 16 points.

Problem 1: In this problem, $\partial = (d/dx)$, and $\delta \in \mathcal{S}^*(\mathbb{R})$ denotes the Dirac delta function.

(a) For $T \in \mathcal{S}^*(\mathbb{R})$, define ∂T, and prove that what you define is a continuous functional on $\mathcal{S}(\mathbb{R})$. (You may use the fact that $\partial : \mathcal{S} \to \mathcal{S}$ is continuous.)

(b) Set $U(x) = x\,[\partial\delta](x)$, and calculate, for $\varphi \in \mathcal{S}$, $\langle U, \varphi\rangle$.

(c) Set $V(x) = x\,\delta(x)$, and calculate, for $\varphi \in \mathcal{S}$, $\langle \partial V, \varphi\rangle$.

Problem 2: We define the functions $\varphi_n \in \mathcal{S}$ by setting $\varphi_n(x) = \frac{x^2}{\sqrt{x^2+1/n}} e^{-x^2}$. Does the sequence converge in $\mathcal{S}$ as $n \to \infty$? If so, to what?

Problem 3: Let H be a Hilbert space and let A be a compact self-adjoint operator on H. Let b be a non-zero real number, and set $f(x) = (x - i\,b)^{-1}$ where i is the imaginary unit. This question concerns different ways of defining $f(A)$.

(a) Noting that f has the MacLaurin expansion $f(x) = (-1/ib)\sum_{n=0}^{\infty}(x/ib)^n$, we define $B_N = (-1/ib)\sum_{n=1}^{N}((1/ib)\,A)^n$. Describe when, if ever, the sequence $(B_N)_{N=1}^{\infty}$ converges in norm in $\mathcal{B}(H)$.

(b) Let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for H consisting of eigenvectors of A, so that $A\,\varphi_n = \lambda_n\,\varphi_n$. Define the operator C_N by setting, for $u \in H$, $C_N\,u = \sum_{n=1}^{N} f(\lambda_n)\,(\varphi_n,\,u)\,\varphi_n$. Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges strongly in $\mathcal{B}(H)$.

(c) Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges in norm in $\mathcal{B}(H)$.

Problem 4: Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n\,\cos(nx) & \text{for } |x| \le R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R, if any, is it the case that $f_n \to 0$ in $\mathcal{S}^*$?