Applied Analysis, Spring 2006: Solutions to the final

Problem 1:

(a)
To ensure that A is a projection: $A^2 = A$.
To ensure that A is orthogonal: $A^* = A$. (Or $\ker(A)^\perp = \mathrm{ran}(A)$, or $||A|| = 1$.)

(b)
(1) is true (see Proposition 9.15).
(2) is true (see Proposition 9.15).
(3) is false. (Example: $A(x_1, x_2, x_3, \dots) = (x_1, x_2/2, x_3/3, \dots)$ on $l^2(\mathbb{N})$.)

(c)
$\varphi_n \to \varphi$ on $\mathcal{S}(\mathbb{R})$ iff $\lim_{n\to\infty} ||\varphi_n - \varphi||_{\alpha,n} = 0$ for all α and n, where
$||\varphi||_{\alpha,n} = \sup_x |(1 + |x|^2)^{n/2} \partial^\alpha \varphi(x)|$.

(d)
$2\pi i \dfrac{\partial \delta}{\partial x_2}$ (for a full solution, see Problem 2 on Midterm 3).

(e)
(1) is not true. (Example: $\chi_{[-1,1]} \in L^1$, but $\hat{\chi}_{[-1,1]} = \frac{2\sin t}{t} \notin L^1$.)
(2) is true.
(3) is true.

(f)
If $(f_n)_{n=1}^\infty$ is a sequence of non-negative measurable functions, then

$$\int \big(\liminf_{n\to\infty} f_n(x)\big)\, d\mu(x) \le \liminf_{n\to\infty} \int f_n(x)\, d\mu(x).$$

(g)

$$f'(\varphi) :\ C(I) \to C(I) :\ h \mapsto \psi \int_0^1 \cos(\varphi(x))\, h(x)\, dx.$$

Problem 2: Since $\cos(2x-2y) = \cos(2x)\,\cos(2y) + \sin(2x)\,\sin(2y)$, we have

$$[Au](x) = \alpha\,\cos(2x)\int_{-\pi}^{\pi}\cos(2y)\,u(y)\,dy + \alpha\,\sin(2x)\int_{-\pi}^{\pi}\sin(2y)\,u(y)\,dy.$$

Setting $\varphi_1(x) = \frac{1}{\sqrt{\pi}}\cos(2x)$ and $\varphi_2(x) = \frac{1}{\sqrt{\pi}}\sin(2x)$, we can write A as

$$Au = \alpha\pi\,\varphi_1\,\langle\varphi_1,\,u\rangle + \alpha\pi\,\varphi_2\,\langle\varphi_2,\,u\rangle. \tag{1}$$

Since φ_1 and φ_2 are orthogonal and normalized, (1) is the spectral decomposition of A.

(a) $\boxed{\text{ran}(A) = \text{span}\{\varphi_1,\,\varphi_2\} = \text{span}\{\cos(2x),\,\sin(2x)\}}$

(b) $\boxed{\sigma(A) = \sigma_{\text{p}}(A) = \{0,\,\pi\alpha\}.}$

To see this, let $(\varphi_n)_{n=3}^{\infty}$ be an ON-basis for $\big(\text{span}\{\varphi_1,\varphi_2\}\big)^{\perp}$ (so that $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H). Then $\alpha\pi$ is an eigenvalue with eigenvectors φ_1 and φ_2, 0 is an eigenvalue with eigenvectors $(\varphi_n)_{n=3}^{\infty}$. If $\lambda\notin\{0,\alpha\pi\}$, then $(A-\lambda I)^{-1}$ is explicitly given by

$$(A-\lambda I)^{-1}\,v = \frac{1}{\alpha\pi-\lambda}\,\langle\varphi_1,\,v\rangle\,\varphi_1 + \frac{1}{\alpha\pi-\lambda}\,\langle\varphi_2,\,v\rangle\,\varphi_2 + \sum_{n=3}^{\infty}\frac{1}{-\lambda}\,\langle\varphi_n,\,v\rangle\,\varphi_n.$$

Note that $||(A-\lambda I)^{-1}|| = \max\left(\frac{1}{|\alpha\pi-\lambda|},\frac{1}{|\lambda|}\right)$, so $(A-\lambda I)^{-1}$ is continuous.

(c) $\boxed{A \text{ is self-adjoint if and only if } \alpha\in\mathbb{R}}$

To see this, note that (1) implies that

$$A^*u = \bar{\alpha}\pi\,\varphi_1\,\langle\varphi_1,\,u\rangle + \bar{\alpha}\pi\,\varphi_2\,\langle\varphi_2,\,u\rangle,$$

so $A^* = A$ if and only if $\bar{\alpha} = \alpha$.

(d) $\boxed{A \text{ is not unitary for any } \alpha.}$

To see this, simply note that, *e.g.* $||A\varphi_3|| = 0 \neq ||\varphi_3||$.

(e) $\boxed{A \text{ is a projection if and only if } \alpha\in\{0,\,1/\pi\}.}$

To see this, note that (1) implies that

$$A^2u = (\alpha\pi)^2\,\varphi_1\,\langle\varphi_1,\,u\rangle + (\alpha\pi)^2\,\varphi_2\,\langle\varphi_2,\,u\rangle = (\alpha\pi)Au.$$

Problem 3: We will prove that there exist finite C and N such that

$$|T\varphi| \leq C \sum_{n,|\alpha|\leq N} ||\varphi||_{\alpha,n}.$$

First we note that

$$(2)\qquad \langle T,\, \varphi\rangle = \lim_{\varepsilon\searrow 0}\int_\varepsilon^\infty \frac{\varphi(x)-\varphi(-x)}{x}\,dx = \underbrace{\lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \frac{\varphi(x)-\varphi(-x)}{x}\,dx}_{=I} + \underbrace{\int_1^\infty \frac{\varphi(x)-\varphi(-x)}{x}\,dx}_{=J}.$$

To bound I, we note that

$$(3)\qquad |I| = \left|\lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \frac{\int_{-x}^{x}\varphi'(t)\,dt}{x}\,dx\right| \leq \int_0^1 \frac{2x\,||\varphi'||_{\text{u}}}{x}\,dx = 2\,||\varphi||_{1,0}.$$

To bound J, we note that

$$(4)\qquad |J| = \left|\int_1^\infty \frac{\varphi(x)-\varphi(-x)}{x}\,dx\right| \leq \int_1^\infty \frac{1}{x^2}\,2\,|x\,\varphi(x)|\,dx = 2\,||\varphi||_{0,1}.$$

Combining (2), (3), and (4), we obtain

$$|\langle T,\, \varphi\rangle| = |I+J| \leq 2\,||\varphi||_{1,0} + 2\,||\varphi||_{0,1}.$$

Problem 4: First note that

$$\left|\frac{t}{1+t^4}\right| \le \frac{1}{3^{1/4}} \le 1 \qquad \forall\, t \in \mathbb{R}.$$

It follows that, for all n and all x,

$$\left|\frac{f_n(x)\,g(x)}{1+(f_n(x))^4}\right| \le |g(x)|.$$

Since $\int_0^\infty |g|\,dx \le ||g||_1 < \infty$, the Lebesgue dominated convergence theorem applies, and

$$\lim_{n\to\infty}\int_0^\infty \frac{f_n(x)\,g(x)}{1+(f_n(x))^4}\,dx = \int_0^\infty \left(\lim_{n\to\infty}\frac{f_n(x)\,g(x)}{1+(f_n(x))^4}\right)dx = \int_0^\infty 0\,dx = 0.$$

Problem 5: If L^p were a Hilbert space, then the parallelogram law would imply that

$$(5) \qquad ||f_1+f_1||_p^2 + ||f_1-f_2||_p^2 - 2||f_1||_p^2 - 2||f_2||_p^2 = 0, \qquad \forall\, f_1, f_2 \in L^p.$$

We need to find f_1 and f_2 such that (5) does not hold. Almost any choices will do; a particularly simple choice is

$$f_1 = \chi_{\Omega_1}, \qquad f_2 = \chi_{\Omega_2},$$

where Ω_1 and Ω_2 are two disjoint sets such that $\mu(\Omega_1) = \mu(\Omega_2) = 1$. Then

$$||f_1+f_2||_p = ||f_1-f_2||_p = \left(\int (\chi_{\Omega_1}^p + \chi_{\Omega_2}^p)\right)^{1/p} = \big(\mu(\Omega_1)+\mu(\Omega_2)\big)^{1/p} = 2^{1/p}$$

Moreover, for $j = 1, 2$,

$$||f_j||_p = \left(\int \chi_{\Omega_j}^p\right)^{1/p} = \mu(\Omega_j)^{1/p} = 1.$$

It follows that

$$||f_1+f_1||_p^2 + ||f_1-f_2||_p^2 - 2||f_1||_p^2 - 2||f_2||_p^2 = 2^{2/p} + 2^{2/p} - 2 - 2,$$

which equals zero if and only if $p = 2$.

Alternative solution: This is a shortcut I hadn't foreseen. It's entirely by the rules, though, so it gets full credit:

We know that for $p \in [1,\infty)$, the dual of $L^p(\mathbb{R}^d)$ is $L^q(\mathbb{R}^d)$, where q is the unique number in $[1,\infty]$ such that $(1/p)+(1/q) = 1$. Since $L^p \ne L^q$, it follows that L^p cannot be a Hilbert space (if it were, then the Riesz representation theorem would state that the dual of L^p is L^p itself).

/// PGM, May 14, 2006