APPM5440 — Applied Analysis: Section exam 2

10:00 – 10:50, Oct. 31, 2016. Closed books.

Important: *Complete problems 1, 2, and 3 in class, and hand your solution in no later than 10:50am. Then complete questions 4 and 5 at home (individual work, no group efforts) and hand the solution in on Friday November 4 at the beginning of class at 10:00am.*

Problem 1: (8 points total, 4 points for each subquestion.)

(a) State the Contraction Mapping Theorem (include a precise definition a contraction).

(b) Does the function $f(x) = x/(1+x)$ acting on the interval $I = [0, \infty)$ satisfy your definition?

Solution:

(a) See book. In grading this problem, I was a stickler about the logic. For full points, you must first posit the existence of $c \in [0,1)$. Then $d(f(x), f(y)) \leq c\, d(x,y) \qquad \forall x, y$.

(b) The given function is **not** a contraction in the sense used in the book. We have

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|} \geq \{\text{Set } y = 0\} \geq \sup_{x \neq 0} \frac{|f(x) - f(0)|}{|x - 0|} = \sup_{x \neq 0} \frac{|x/(1+x)|}{|x|} = \sup_{x \neq 0} \frac{1}{|1+x|} = 1.$$

(Some of you proved rigorously that $|f(x) - f(y)| < |x - y|$. This is indeed a kind of contraction, so this earned 1 point. However, if you proved that $|f(x) - f(y)| \leq |x - y|$, then this merited 0 points since this is not a contraction.)

Problem 2: (8 points) Set $I = [0,1]$ and define for $f \in C(I)$ the function $\varphi(f) = \sup_{x \in I} |x\, f(x)|$. Please motivate your answers to all questions.

(a) (2p) Prove that $\varphi(f+g) \leq \varphi(f) + \varphi(g)$ for all $f, g \in C(I)$.

(b) (3p) Let $\|f\|_{\rm u} = \sup_{x \in I} |f(x)|$ denote the standard norm on $C(I)$. Determine the number

$$M = \sup_{f \in C(I),\, f \neq 0} \frac{\varphi(f)}{\|f\|_{\rm u}}.$$

(c) (3p) Let $\|f\|_{\rm u} = \sup_{x \in I} |f(x)|$ denote the standard norm on $C(I)$. Determine the number

$$m = \inf_{f \in C(I),\, f \neq 0} \frac{\varphi(f)}{\|f\|_{\rm u}}.$$

Solution:

(a) This is straight-forward.

(b) First, note that

$$M = \sup_{f \neq 0} \frac{\varphi(f)}{\|f\|_{\rm u}} = \sup_{f \neq 0} \frac{\sup_x |xf(x)|}{\|f\|_{\rm u}} \leq \sup_{f \neq 0} \frac{\sup_x |f(x)|}{\|f\|_{\rm u}} = \sup_{f \neq 0} \frac{\|f\|_{\rm u}}{\|f\|_{\rm u}} = 1.$$

Next, we have

$$M = \sup_{f \neq 0} \frac{\varphi(f)}{\|f\|_{\rm u}} \geq \{\text{Set} f = 1\} \geq \frac{\varphi(1)}{\|1\|_{\rm u}} = \frac{\sup_x |x|}{\sup_x 1} = \frac{1}{1} = 1.$$

So $M = 1$.

(c) Consider the functions

$$f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n) \\ 0 & x \in [1/n, 1]. \end{cases}$$

Then clearly $\|f_n\|_{\rm u} = 1$. Moreover,

$$\varphi(f_n) = \sup_{x \in [0,1]} |xf_n(x)| = \sup_{x \in [0,1/n]} |x(1-nx)| \leq \sup_{x \in [0,1/n]} |x| = 1/n.$$

Putting everything together, we find that

$$m = \inf_{f \in C(I),\, f \neq 0} \frac{\varphi(f)}{\|f\|_{\rm u}} \leq \inf_{n = 1,2,3,\ldots} \frac{\varphi(f_n)}{\|f_n\|_{\rm u}} \leq \inf_{n=1,2,3,\ldots} \frac{1/n}{1} = 0.$$

Since m must be non-negative, we find $m = 0$.

Note: In the exam, φ was defined via $\varphi(f) = \sup_{x \in I} \|x\, f(x)\|$. The norm around $x\, f(x)$ was a typo, and I regret if this caused confusion. Note that it does not really change anything, though. The quantity "$x\, f(x)$" is a scalar, and the "norm" of a scalar is of course the absolute value.

Problems 3 and 4: (8 points) Answer TRUE or FALSE for each of the questions.

(a) Set $I = [0, \infty)$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is equicontinuous, bounded, and does not have a convergent subsequence.

(b) Set $I = [0, 1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is equicontinuous, and does not have a convergent subsequence.

(c) Set $I = [0, 1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is bounded, and does not have a convergent subsequence.

(d) Set $I = [0, 1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is bounded, equicontinuous, and does not have a convergent subsequence.

(e) Set $I = [0, 1]$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ that converges pointwise. Furthermore, suppose that $\mathrm{Lip}(f_n) \le 2$ for every n. Then the numbers $I_n = \int_0^1 f_n(x)\,dx$ form a convergent sequence in $\mathbb{R}$.

Solution:

(a) TRUE. For an example, consider

$$f_n(x) = \begin{cases} 0 & x \in [0, n-1], \\ 1 - |x - n| & x \in (n-1, n+1), \\ 0 & x \in [n+1, \infty). \end{cases}$$

In other words, (f_n) is a sequence of "tent functions" moving off to the right. Then (f_n) is bounded and equicontinuous. Since $\|f_n - f_m\| \ge |f_n(n) - f_m(n)| = |1 - 0| = 1$ whenever $m \ne n$, we see that (f_n) cannot have a convergent subsequence.

(b) TRUE. Set $f_n = n$. Then (f_n) is equicontinuous, since $\mathrm{Lip}(f_n) = 0$ for every n. Moreover $\|f_n - f_m\|_{\mathrm{u}} = |n - m|$ so (f_n) cannot have a convergent subsequence.

(c) TRUE. Set $f_n(x) = x^n$. Then (f_n) is bounded since $\|f_n\|_{\mathrm{u}} = 1$. But (f_n) converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & x \in [0, 1), \\ 1 & x = 1. \end{cases}$$

If (f_n) had a convergent subsequence, then this subsequence would have to have f as its limit point too, which is impossible since the uniform limit of continuous functions must be continuous.

(d) FALSE. The AA theorem says that if (f_n) is a bounded equicontinuous sequence of functions on a compact set I, then (f_n) must have a uniformly convergent subsequence.

(e) TRUE. (No motivation required here, but recall from Problem 1 on HW6 that (f_n) must be uniformly convergent. This means that the integrals must converge too.)

Note: It is NOT sufficient for (a), (b), and (c) to merely point out how the AA theorem is violated. This simply says that the AA theorem does not preclude the existence of a sequence like the one described. You need to do more to affirmatively prove existence. For instance, you can give an example, like the ones above.

In grading problem 4, I did not give credit for a correct answer backed up by an invalid motivation.

Problem 5: (8 points) Let T be the integral operator that takes a function $u \in C(\mathbb{R})$ and maps it to

$$[Tu](x) = \int_0^x \big((u(y))^3 + \cos(u(y))\big)\, dy + 1, \qquad x \in \mathbb{R}.$$

(a) (5p) Let δ and B be positive numbers and set

$$\Omega = \{u \in C([0,\delta]) : \ \|u\|_{\mathrm{u}} \le B\}.$$

Prove that if B and δ are small enough, then $T : \Omega \to \Omega$ is a *contraction*, as defined in Definition 3.1 in the text book (with respect to the uniform norm, of course). Your numbers B and δ do not need to be optimal, but if your estimate is unnecessarily crude, then some small number of points might be deducted.

(b) (3p) Prove that the operator T is not a contraction as a map from $C([0,\delta])$ to $C([0,\delta])$ for any $\delta > 0$.

Solution:

(a) First we need to ensure that $Tu \in \Omega$ whenever $u \in \Omega$. We have

$$\|Tu\| = \sup_x \left| \int_0^x \big(u(y)^3 + \cos(u(y))\big)\, dy + 1 \right| \le \sup_x \left(\int_0^x \big(B^3 + 1\big)\, dy + 1 \right) = \delta(B^3+1) + 1.$$

So we must have: $\boxed{\delta(B^3+1)+1 \le B}$

Next we check if T is a contraction. Set $f(t) = t^3 + \cos(t)$, so that $[Tu](x) = \int_0^x f(u(y))\, dy + 1$. Then

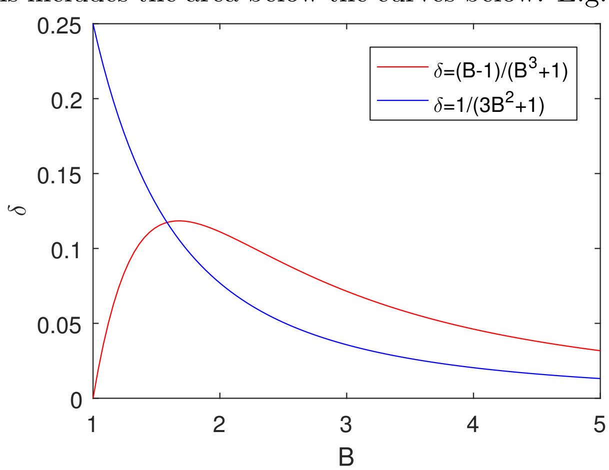
$$[Tu](x) - [Tv](x) = \left(\int_0^x f(u(y))\, dy + 1 \right) - \left(\int_0^x f(v(y))\, dy + 1 \right) = \int_0^x \big(f(u(y)) - f(v(y))\big)\, dy.$$

Now observe that $f'(t) = 3t^2 + \sin(t)$. If $|t| \le B$, then $|f'(t)| \le 3B^2 + 1$. It follows that if $u, v \in \Omega$, then $|f(u(y)) - f(v(y))| \le (3B^2+1)|u(y) - v(y)|$. Now, for $u, v \in \Omega$,

$$\|Tu - Tv\| = \sup_x \left| \int_0^x \big(f(u(y)) - f(v(y))\big)\, dy \right| \le \sup_x \left| \int_0^x (3B^2+1)|u(y) - v(y)| dy \right| \le \delta(3B^2+1)\|u - v\|.$$

We see that T is a contraction on Ω if: $\boxed{\delta(3B^2+1) < 1}$

The set of possible solutions includes the area below the curves below. E.g. $B = 2$ and $\delta = 0.05$ works.



(b) Fix $\delta > 0$. Set $v = \pi/2$ and $u_n = \pi/2 + 2\pi n$ for $n = 1, 2, 3, \ldots$. Then $Tv = 1 + \delta(\pi/2)^3$ and $Tu_n = 1 + \delta(\pi/2 + 2\pi n)^3$. It follows that

$$\frac{\|Tu_n - Tv\|}{\|u_n - v\|} = \frac{\delta(\pi/2 + 2\pi n)^3 - \delta(\pi/2)^3}{2\pi n}.$$

As $n \to \infty$, this ratio goes to infinity, so T cannot be a contraction.