APPM5440 — Applied Analysis: Section exam 2
10:00 – 10:50, Oct. 31, 2016. Closed books.

Important: *Complete problems 1, 2, and 3 in class, and hand your solution in no later than 10:50am. Then complete questions 4 and 5 at home (individual work, no group efforts) and hand the solution in on Friday November 4 at the beginning of class at 10:00am.*

Problem 1: (8 points total, 4 points for each subquestion.)

(a) State the Contraction Mapping Theorem (include a precise definition a contraction).

(b) Does the function $f(x) = x/(1+x)$ acting on the interval $I = [0,\infty)$ satisfy your definition?

Problem 2: (8 points) Set $I = [0,1]$ and define for $f \in C(I)$ the function $\varphi(f) = \sup_{x\in I} |x\, f(x)|$. Please motivate your answers to all questions.

(a) (2p) Prove that $\varphi(f+g) \leq \varphi(f) + \varphi(g)$ for all $f, g \in C(I)$.

(b) (3p) Let $\|f\|_{\mathrm{u}} = \sup_{x\in I} |f(x)|$ denote the standard norm on $C(I)$. Determine the number

$$M = \sup_{f\in C(I),\, f\neq 0} \frac{\varphi(f)}{\|f\|_{\mathrm{u}}}.$$

(c) (3p) Let $\|f\|_{\mathrm{u}} = \sup_{x\in I} |f(x)|$ denote the standard norm on $C(I)$. Determine the number

$$m = \inf_{f\in C(I),\, f\neq 0} \frac{\varphi(f)}{\|f\|_{\mathrm{u}}}.$$

Problem 3: (8 points) Answer TRUE or FALSE for each of the questions. No motivation necessary. In grading this problem, 2 points will be deducted for each wrong answer.

(a) Set $I = [0,\infty)$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is equicontinuous, bounded, and does not have a convergent subsequence.

(b) Set $I = [0,1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is equicontinuous, and does not have a convergent subsequence.

(c) Set $I = [0,1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is bounded, and does not have a convergent subsequence.

(d) Set $I = [0,1]$. There exists a sequence $(f_n)_{n=1}^{\infty}$ in $C(I)$ that is bounded, equicontinuous, and does not have a convergent subsequence.

(e) Set $I = [0,1]$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ that converges pointwise. Furthermore, suppose that $\mathrm{Lip}(f_n) \leq 2$ for every n. Then the numbers $I_n = \int_0^1 f_n(x)\,dx$ form a convergent sequence in $\mathbb{R}$.

(Of course, we always use the uniform norm in $C(I)$.)

Take home exam below.

Problem 4: (8 points) Redo Problem 3(a,b,c,d), but now provide motivations for your answers. For instance, you may cite a theorem in the syllabus, and describe how it provides a definitive answer to a question. (Observe that in order to prove that something exists, it is not sufficient to say that existence is not prohibited by any given theorem.)

Note: You do *not* need to stick to the answers you gave in the in-class exam. If you want to change your mind, then please do so. Problems 3 and 4 will be graded independently, and there will be no "extra" penalty for inconsistent answers.

Problem 5: (8 points) Let T be the integral operator that takes a function $u \in C(\mathbb{R})$ and maps it to

$$[Tu](x) = \int_0^x \big((u(y))^3 + \cos(u(y))\big)\, dy + 1, \qquad x \in \mathbb{R}.$$

(a) (5p) Let δ and B be positive numbers and set

$$\Omega = \{u \in C([0,\delta]) : \ \|u\|_{\mathrm{u}} \le B\}.$$

Prove that if B and δ are small enough, then $T : \Omega \to \Omega$ is a *contraction,* as defined in Definition 3.1 in the text book (with respect to the uniform norm, of course). Your numbers B and δ do not need to be optimal, but if your estimate is unnecessarily crude, then some small number of points might be deducted.

(b) (3p) Prove that the operator T is not a contraction as a map from $C([0,\delta])$ to $C([0,\delta])$ for any $\delta > 0$.