Hints for homework set 12 — APPM5440 — Fall 2016

Problem 1: Let X denote the linear space of polynomials of degree 2 or less on $I = [0,\,1]$. For $f \in X$, set $||f|| = \sup_{x\in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1, 2, 3, 4$. Prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Solution:

You can easily prove that all the φ_j's are linear.

To prove that each φ_j is continuous, simply invoke the theorem that any linear map on a finite dimensional space is continuous. For fun, let's work it out explicitly for $j = 1, 2$:

$$|\varphi_1(f)| \leq \int_0^1 |f(x)|\,dx \leq \int_0^1 ||f||\,dx = ||f||.$$
$$|\varphi_2(f)| = |f(0)| \leq ||f||.$$

Proving directly that φ_3 and φ_4 are bounded takes a little more work.

To prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis, we first observe that since X has dimension three, we know that X^* also has dimension three. So all we need to prove is that the set is linearly independent. Suppose

$$c_1\,\varphi_1 + c_2\,\varphi_2 + c_3\,\varphi_3 = 0.$$

In other words, for every $f \in X$, we must have $c_1\,\varphi_1(f) + c_2\,\varphi_2(f) + c_3\,\varphi_3(f) = 0$. By plugging in $f = 1$, $f = x$, $f = x^2$, we get three equations for c_1, c_2, and c_3,

$$\begin{aligned} f = 1 \quad &\Rightarrow \quad c_1 + c_2 = 0,\\ f = x \quad &\Rightarrow \quad (1/2)c_1 + c_3 = 0,\\ f = x^2 \quad &\Rightarrow \quad (1/3)c_1 + c_3 = 0. \end{aligned}$$

It is easy to show that the only solution is $c_1 = c_2 = c_3 = 0$.

To prove that $\{\varphi_1,\,\varphi_2,\,\varphi_4\}$ does not form a basis, we will prove that they are linearly dependent. Suppose

$$c_1\,\varphi_1 + c_2\,\varphi_2 + c_4\,\varphi_4 = 0.$$

By plugging in $f = 1$, $f = x$, $f = x^2$, we get three equations for c_1, c_2, and c_4,

$$\begin{aligned} f = 1 \quad &\Rightarrow \quad c_1 + c_2 = 0,\\ f = x \quad &\Rightarrow \quad (1/2)c_1 + c_4 = 0,\\ f = x^2 \quad &\Rightarrow \quad (1/3)c_1 + (2/3)c_4 = 0. \end{aligned}$$

This system has infinitely many solutions. For any $t \in \mathbb{R}$, the triple $\{c_1 = t,\, c_2 = -t,\, c_4 = -t/2\}$ is a solution. In particular, for $t = 1$ we find that

$$\varphi_1 - \varphi_2 - \frac{1}{2}\varphi_4 = 0.$$

Problem 2: Let $X = \ell^2$. Recall from class that every $\varphi \in X^*$ is of the form $\varphi(x) = \sum x_n\, y_n$ for some $y \in X$. Set $D = \{x \in \ell^2 : ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in ℓ^2. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^{\infty} \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Set $Y = \ell^3$. What is Y^*? Prove that the weak closure of the surface of the unit ball in ℓ^3 is the closed unit ball in ℓ^3.

Solution:

Fix $x \in X$ such that $||x|| < 1$. For $n = 1, 2, 3, \dots$, define α_n via

$$\alpha_n = \sqrt{1 - \sum_{j \neq n} x_j^2}.$$

Since $||x|| < 1$, we know that $0 < \alpha_n \leq 1$ for every n. Then define $x^{(n)}$ as the sequence obtained by swapping x_n for α_n. In other words

$$x^{(n)} = (x_1,\, x_2,\, \dots,\, x_{n-2},\, x_{n-1},\, \alpha_n,\, x_{n+1},\, x_{n+2},\, \dots).$$

We find that

$$||x^{(n)}||^2 = \alpha_n^2 + \sum_{j \neq n} x_j^2 = 1$$

so $x^{(n)} \in D$ for every n.

It remains to show that $x^{(n)} \rightharpoonup x$. Fix $y \in X$. Then

$$(x - x_n, y) = (x_n - \alpha_n)\, y_n.$$

Since $||y_n|| \to 0$ and $|x_n - \alpha_n| \leq |x_n| + |\alpha_n| \leq 2$, we find that

$$\lim_{n \to \infty} (x - x_n, y) = 0.$$

In treating the set Y, we first recall that the dual of ℓ^p for $p \in (1, \infty)$ is the set ℓ^q where $1/p + 1/q = 1$. When $p = 3$, we find $q = 3/2$. In other words, any $\varphi \in Y^3$ takes the form

$$\varphi(x) = \sum_{n=1}^{\infty} x_n\, y_n,$$

where $y \in \ell^{3/2}$. The density argument now follows pretty much the same lines as it did for $X = \ell^2$. Given x such that $||x|| < 1$, pick α_n so that

$$\alpha_n = \left(1 - \sum_{j \neq n} |x_j|^3\right)^{1/3},$$

set

$$x^{(n)} = (x_1,\, x_2,\, \dots,\, x_{n-2},\, x_{n-1},\, \alpha_n,\, x_{n+1},\, x_{n+2},\, \dots),$$

show that $||x^{(n)}|| = 1$, and then that $x^{(n)} \rightharpoonup x$.

Problem 3: Consider the space $X = \ell^2$ and let $T \in \mathcal{B}(X)$ be a compact operator such that $\ker(T) = \{0\}$. Prove that $\operatorname{ran}(T)$ is not closed.

Solution:

We prove this by contradiction. Suppose that $\ker(T) = \{0\}$ and that $\operatorname{ran}(T)$ is closed. We know from the closed range theorem that then there exists a positive c such that

$$\|Tx\| \geq c\|x\|, \qquad \forall x \in X.$$

Let $(e_n)_{n=1}^{\infty}$ be the canonical basis vectors

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \dots) \\ e_2 &= (0, 1, 0, 0, \dots) \\ e_3 &= (0, 0, 1, 0, \dots) \\ &\vdots \end{aligned}$$

Observe that if $m \neq n$, then $\|e_m - e_n\| = \sqrt{2}$. It now follows that $(Te_n)_{n=1}^{\infty}$ cannot have a convergent subsequence since, for $m \neq n$, we have $\|Te_m - Te_n\| \geq c\sqrt{2}$. It follows that T cannot be compact.