Homework set 12 — APPM5440 — Fall 2016

Problem 1: Let X denote the linear space of polynomials of degree 2 or less on $I = [0,\,1]$. For $f \in X$, set $||f|| = \sup_{x\in I} |f(x)|$. For $f \in X$, define

$$\varphi_1(f) = \int_0^1 f(x)\,dx, \quad \varphi_2(f) = f(0), \quad \varphi_3(f) = f'(1/2), \quad \varphi_4(f) = f'(1/3).$$

Prove that $\varphi_j \in X^*$ for $j = 1,2,3,4$. Prove that $\{\varphi_1,\,\varphi_2,\,\varphi_3\}$ forms a basis for X^*. Prove that $\{\varphi_1, \varphi_2, \varphi_4\}$ does not form a basis for X^*.

Hint: Any $f \in X$ can be written $f(x) = a + b\,x + c\,x^2$ for unique a, b, and c.

Problem 2: Let $X = \ell^2$. Recall from class that every $\varphi \in X^*$ is of the form $\varphi(x) = \sum x_n\,y_n$ for some $y \in X$. Set $D = \{x \in \ell^2 : ||x|| = 1\}$. Prove that the weak closure of D is the closed unit ball in ℓ^2. (Hint: To prove that the closed unit ball is contained in the weak closure of D, you can for any element x such that $||x|| < 1$ explicitly construct a sequence $(x^{(n)})_{n=1}^\infty \subset D$ that weakly converges to x, such that $||x^{(n)}|| = 1$.)

Set $Y = \ell^3$. What is Y^*? Prove that the weak closure of the surface of the unit ball in ℓ^3 is the closed unit ball in ℓ^3.

Problem 3: Consider the space $X = \ell^2$ and let $T \in \mathcal{B}(X)$ be a compact operator such that $\ker(T) = \{0\}$. Prove that $\mathrm{ran}(T)$ is not closed.