Solutions to homework set 8 — APPM5440 — Fall 2016

3.6: Set $X = C([-a, a])$ and define on X the operator
$$[Fu](x) = \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} u(y) \, dy + 1.$$
Then the given equation can be formulated as a fixed point problem $u = F(u)$. We find that
$$||F(u) - F(v)|| = \sup_{x \in [-a,a]} \left| \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} (u(y) - v(y)) \, dy \right| \leq \cdots \leq \frac{2 \arctan(a)}{\pi} ||u - v||$$
(you should be able to fill in the missing steps). The contraction mapping principle now proves that the equation has a unique solution in X.

To prove that the solution is positive, note that if $u(x) \geq 0$ for all x, then $[Tu](x) \geq 0$ for all x. Combine this fact with the fact that f is non-negative, and that
$$u = (I - T)^{-1} f = \sum_{n=0}^{\infty} T^n f.$$

For the case $a = \infty$, note that uniqueness cannot hold. If u is a solution, then so is any shift of u.

3.7: Set $X = \{u \in C(I) : u(0) = u(1) = 0\}$. Convolve given BVP with the Green's function $g(x, y)$ (defined by eqn (3.22)) and obtain
$$(1) \qquad\qquad\qquad\qquad u + \lambda T u = h,$$
where T is the operator on X given by
$$[Tu](x) = \int_{0}^{1} g(x, y) \sin(u(y)) \, dy,$$
and
$$h(x) = \int_{0}^{1} g(x, y) f(y) \, dy.$$
Note that $Tf \in X$, so (1) is an equation on X. Now
$$||Tu - Tv|| \leq \sup_{x} \int_{0}^{1} |g(x, y)| \, |\sin(u(y)) - \sin(v(y))| \, dy.$$
Use that
$$|\sin(u(y)) - \sin(v(y))| \leq |u(y) - v(y)| \leq ||u - v||$$
to obtain
$$||Tu - Tv|| \leq \beta ||u - v||,$$
where
$$\beta = \sup_{x} \int_{0}^{1} |g(x, y)| \, dy.$$
We see that if $\lambda < 1/\beta$, equation (1) has a unique solution in X (by the contraction mapping theorem).

Problem 4: Consider the integral equation

(*)
$$u(x) = \pi^2 \sin(x) + \frac{3}{2} \int_0^{\cos(x)} |x - y|\, u(y)\, dy.$$

Prove that (*) has a unique solution in $C([0,1])$.

Solution: Set $X = C([0,1])$ and define the operator T on X by

$$[Tu](x) = \frac{3}{2} \int_0^{\cos(x)} |x - y|\, u(y)\, dy.$$

We find that

$$\|Tu - Tv\| \le \sup_x \frac{3}{2} \int_0^{\cos(x)} |x - y|\, |u(y) - v(y)|\, dy$$

$$\le \sup_x \frac{3}{2} \int_0^1 |x - y|\, |u(y) - v(y)|\, dy$$

$$\le \sup_x \frac{3}{2} \int_0^1 |x - y|\, dy\, \|u - v\|.$$

Now

$$\sup_x \int_0^1 |x - y|\, dy = \sup_x \left(\frac{1}{2}x^2 + \frac{1}{2}(1 - x)^2\right) = \frac{1}{2},$$

so

$$\|Tu - Tv\| \le \frac{3}{4}\|u - v\|.$$

Since T is a contraction, the equation

$$(I - T)\, u = f$$

has a unique solution for every f. (In particular, for $f(x) = \pi^2 \sin(x)$.)

Problem 1: Let X be a set with infinitely many members. We define a collection $\mathcal{T}$ of subsets of X by saying that a set $\Omega \in \mathcal{T}$ if either $\Omega^c = X \backslash \Omega$ is finite, or if Ω is the empty set. Verify that $\mathcal{T}$ is a topology on X. This topology is called the "co-finite" topology on X. Describe the closed sets.

Solution: The verification should be straight-forward. The closed sets are the finite sets, and the entire set.

Problem 2: Let X denote a finite set, and let $\mathcal{T}$ be a metrizable topology on X. Prove that $\mathcal{T}$ is the discrete topology on X.

Solution: Enumerate the points in X so that $X = \{x_n\}_{n=1}^N$. Set
$$\varepsilon_n = \min\{d(x_n, x_m) : \ m \neq n\}, \quad \text{for } n = 1, 2, \ldots, N.$$
Since the minimum is taken over a finite set of positive numbers, $\varepsilon_n > 0$, and it follows that the set $B_{\varepsilon_n/2}(x_n) = \{x_n\}$, must be open. Since any union of open set must itself be open, it follows that all subsets of X are open.

Problem 3: Consider the set $X = \{a, b, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Is $\mathcal{T}$ a topology? Is $\mathcal{T}$ a metrizable topology?

Solution: Yes it is a topology (and union or intersection of the given sets is itself a member of the set.

No, it cannot be metrizable. If it were, then the argument given in Problem 2 would demonstrate that $\{b\}$ would have to be an open set, and it is not.