Solutions to homework set 6 — APPM5440 — Fall 2016

2.10: Let A denote the set of functions in $C(\mathbb{R}^n)$ that vanish at infinity. That $A = \overline{C_{\mathrm{c}}(\mathbb{R}^n)}$ follows from the following two claims:

- Claim 1: $C_{\mathrm{c}}(\mathbb{R}^n)$ is dense in A.
- Claim 2: A is closed.

Proof of Claim 1: Fix an $f \in A$. We need to prove that for any $\varepsilon > 0$, there exists a $g \in C_{\mathrm{c}}$ such that $||f - g||_{\mathrm{u}} < \varepsilon$. Fix $\varepsilon > 0$. Set $R = \sup\{|x| : |f(x)| \geq \varepsilon\}$ (so that $|f(x)| \leq \varepsilon$ when $|x| \geq R$). Set for $x \in \mathbb{R}^n$

$$\varphi_R(x) = \begin{cases} 1 & |x| \in [0, R), \\ 1 + R - |x| & |x| \in [R, R+1], \\ 0 & |x| \in (R, \infty), \end{cases}$$

and set $g = f\,\varphi_R$. Then $g \in C_{\mathrm{c}}$, and $||f - g||_{\mathrm{u}} < \varepsilon$.

Proof of Claim 2: We will prove that $C(I)\backslash A$ is open. Fix an $f \in C(I)\backslash A$. Then for some $\varepsilon > 0$, there exist $(x_j)_{j=1}^{\infty} \in \mathbb{R}^n$ such that $|f(x_j)| \geq \varepsilon$ for all j, and $|x_j| \to \infty$. Then if $h \in C(I)$, and $||f - h|| < \varepsilon/2$, we find that

$$|h(x_j)| = |f(x_j) + (h(x_j) - f(x_j))| \geq |f(x_j)| - |h(x_j) - f(x_j)| > \varepsilon/2,$$

and so $h \in C(I)\backslash A$. It follows that $B_{\varepsilon/2}(f) \subseteq C(I)\backslash A$.

2.11: Set $g_n = f_n - f$. Then for every $x \in I$, $g_n(x) \searrow 0$. We need to prove that g_n converges uniformly to 0.

Since $g_n(x) \searrow 0$ for every x, $(||g_n||_{\mathrm{u}})_{n=1}^{\infty}$ is a decreasing sequence. Set $\alpha = \lim_{n\to\infty} ||g_n||_{\mathrm{u}}$. If $\alpha = 0$, then $g_n \to 0$ uniformly. Assume $\alpha \neq 0$. Then for each $n = 1, 2, \dots$, there exists a point $x_n \in I$ such that $g_n(x_n) \geq \alpha$ (since g_n is continuous on a compact set). Since I is compact, there exists an $x \in I$ and a subsequence n_j such that $x_{n_j} \to x$. Since $g_n(x) \searrow 0$, there exists an N such that $g_N(x) < \alpha/2$. Since g_N is continuous at x, there exists an $\varepsilon > 0$ such that $g_N(y) < 3\alpha/4$ for all $y \in B_\varepsilon(x)$. But then $g_n(y) < 3\alpha/4$ for all $n \geq N$ (since $g_n(y) \leq g_N(y)$ when $n \geq N$). This contradicts the claims that $g_{n_j}(x_{n_j}) \geq \alpha$, and $x_j \to x$ as $j \to \infty$.

A more elegant solution (that is perhaps harder to think of?): Fix $\varepsilon > 0$. Set $G_n = \{x \in I : |f(x) - f_n(x)| < \varepsilon\}$. Then:

(1) Each G_n is open since both f and f_n are continuous (with $g_n = f_n - f$ we have $G_n = g_n^{-1}(B_\varepsilon(0))$).
(2) Since for any x, $|f(x) - f_n(x)| \geq |f(x) - f_{n+1}(x)|$ we have $G_n \subseteq G_{n+1}$.
(3) $\bigcup_{n=1}^{\infty} G_n = I$. (Every x belongs to some G_n since $f_n(x) \to f(x)$.)

Since I is compact and $\{G_n\}_{n=1}^{\infty}$ is an open cover, there is a finite N such that $I = \bigcup_{n=1}^{N} G_n = G_N$. This means that for $n \geq N$, we have $||f_n - f|| \leq \varepsilon$.

2.12: Fix an $x \in [0, 1]$. Fix an $\varepsilon > 0$. Since $\Omega = \{f_n\}$ is equicontinuous, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon/2$. Now, if $|x - y| < \delta$, then

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq \limsup_{n\to\infty} \varepsilon/2 = \varepsilon/2.$$

2.14: Set

$$e(t) = |u(t) - u_0|.$$

Then e satisfies

$$e(t) = |u(t) - u(t_0)| = \left| \int_{t_0}^{t} f(s, u(s))\, ds \right| \leq \int_{t_0}^{t} |f(s, u(s))|\, ds. \tag{1}$$

Now use that

$$\begin{aligned} |f(s,u(s))| &= |(f(s,u(s)) - f(s,u_0)) + f(s,u_0)| \\ &\leq |f(s,u(s)) - f(s,u_0)| + |f(s,u_0)| \\ &\leq K|u(s) - u_0| + M \\ &= K\, e(s) + M. \end{aligned} \tag{2}$$

Inserting (2) into (1) we find that

$$e(t) \leq M|t - t_0| + \int_{t_0}^{t} K\, e(s)\, ds.$$

A direct application of Grönwall's inequality results in

$$e(t) \leq M\, |t - t_0|\, e^{K\, |t - t_0|}.$$

For the last part of the problem, the exact solution of the given ODE is $u(t) = u_0\, e^{K(t-t_0)}$, and so

$$|u(t) - u_0| = |u_0|\, |e^{K(t-t_0)} - 1| \leq |u_0|\, K|t - t_0|\, e^{K|t-t_0|},$$

since $|e^{\alpha} - 1| \leq |\alpha| e^{|\alpha|}$ for all real α. Since in this example $f(t,u) = K\, u$, and $M = K\, |u_0|$, we see that the given solution satisfies the bound we proved.

Problem 1: Fix $\varepsilon > 0$. Set $\delta = \varepsilon/3C$. Then the Lipschitz condition implies that for any n,

$$d(x,y) < \delta \qquad \Rightarrow \qquad d(f_n(x), f_n(y)) < \varepsilon/3. \tag{3}$$

Since X is compact, there exist points $\{x_j\}_{j=1}^{J}$ such that $X = \bigcup_{j=1}^{J} B_\delta(x_j)$. Since $f_n(x_j) \to f(x_j)$ for every j, and there are only finitely many points x_j, we can pick an N such that

$$m, n \geq N \qquad \Rightarrow \qquad |f_n(x_j) - f_m(x_j)| < \varepsilon/3, \qquad j = 1,\, 2,\, 3,\, \ldots,\, J. \tag{4}$$

Pick any $x \in X$. Suppose $m, n \geq N$. Pick x_j such that $d(x, x_j) < \delta$. Then

$$|f_m(x) \;-\; f_n(x)| \quad \leq \quad \underbrace{|f_m(x) - f_m(x_j)|}_{\leq \varepsilon/3} + \underbrace{|f_m(x_j) - f_n(x_j)|}_{< \varepsilon/3} + \underbrace{|f_n(x_j) - f_n(x)|}_{\leq \varepsilon/3} \quad < \quad \varepsilon.$$

The first and the last terms are bounded by (3) and the middle one by (4).