Homework set 3 — APPM5440, Fall 2016

From the textbook: 1.17, 1.18, 1.20, 1.22, 1.27.

Solution for 1.20: Show that an NLS X is complete iff it is the case that every absolutely convergent sum converges.

Assume that X is complete: Let (x_n) be a sequence such that $\sum_{n=1}^{\infty} ||x_n|| < \infty$. Set

$$s_m = \sum_{n=1}^{m} x_n.$$

We need to show that (s_m) converges in X. We will do this by showing that (s_m) is Cauchy, and then the completeness of X will imply convergence. Fix $\varepsilon > 0$. Then pick an N such that

$$\sum_{n=N+1}^{\infty} ||x_n|| < \varepsilon.$$

Now suppose $N \leq m < k$. Then

$$||s_m - s_k|| = ||\sum_{n=m+1}^{k} x_n|| \leq \sum_{n=m+1}^{k} ||x_n|| < \varepsilon.$$

Assume that every absolutely convergent sum converges: Let (y_m) be a Cauchy sequence in X. Pick a subsequence (y_{m_j}) such that $||y_{m_j} - y_{m_{j-1}}|| \leq 2^{-j}$. Set

$$x_1 = y_{m_1}$$

and then set for $n = 2, 3, 4, \ldots$

$$x_n = y_{m_n} - y_{m_{n-1}}.$$

Observe that

$$\sum_{n=1}^{\infty} ||x_n|| \leq ||x_1|| + \sum_{n=2}^{\infty} \frac{1}{2^n} < \infty.$$

Next note that

$$y_{m_j} = \sum_{n=1}^{j} x_n.$$

By assumption, we then know that y_{m_j} converges to some limit point $y \in X$. All that remains is to show that (y_m) also converges to y. Fix $\varepsilon > 0$. Pick N such that

$$m, k \geq N \qquad \Rightarrow \qquad ||y_m - y_k|| < \varepsilon/2.$$

Then pick m_j such that $||y - y_{m_j}|| < \varepsilon/2$ and $m_j \geq N$. Then

$$m \geq N \qquad \Rightarrow \qquad ||y - y_m|| \leq ||y - y_{m_j}|| + ||y_{m_j} - y_m|| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Solution for 1.27: Suppose x_n does not converge to x. Then there exists an $\varepsilon > 0$ and a subsequence such that $d(x_{n_j}, x) > \varepsilon$. Since the space is compact, (x_{n_j}) has a convergent subsequence. But then by assumption, this subsequence must converge to x, which is impossible since $d(x_{n_j}, x) > \varepsilon$ for all j.

Problem 1: We define a subset Ω of $\mathbb{R}$ via

$$\Omega = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left[\tfrac{1}{n+1/2},\, \tfrac{1}{n}\right] \right).$$

Prove that Ω is compact.

Outline of solution: Ω is totally bounded since any bounded subset of $\mathbb{R}$ is. That Ω is complete follows from the fact that $\mathbb{R}$ is complete, if we can only prove that Ω is closed. An easy way to do this is to write $\Omega^{\rm c}$ as an infinite union of open sets.

Problem 2: Consider our recurring example of the metric space $\mathbb{Q}$ (with the standard metric), and its subset $\Omega = \{q \in \mathbb{Q} :\ q^2 < 2\}$.

(a) Prove the Ω is both open and closed in $\mathbb{Q}$.

(b) Ω is bounded. Does the claim in (a) imply that Ω is compact? If yes, then motivate, if not, then decide whether Ω is in fact compact.

Outline of solution: For (a), simply use the definition. To prove that Ω is open, pick a point $q \in \Omega$, and then construct an ε ball around it entirely contained in Ω. Then prove that $\Omega^{\rm c}$ is open analogously. For (b), note that (a) does not imply that Ω is compact since the underlying space, $\mathbb{Q}$ is not complete. In fact, Ω is not compact. An easy way to prove this is to prove that Ω is to construct a sequence in Ω that does not have a convergent subsequence.

Problem 3: Let X be an infinite set equipped with the discrete metric. Decide which subsets of X (if any) are compact.

Solution: A set Ω in (X, d) is compact iff it is finite. Suppose that Ω is finite, $\Omega = \{x_j\}_{j=1}^{n}$. Then Ω is closed (any set is) and it is also totally bounded since for any ε, the sets $\{B_\varepsilon(x_j)\}_{j=1}^{n}$ cover Ω. Conversely, suppose that Ω is infinite. Then $\{B_{1/2}(x)\}_{x\in\Omega} = \{\{x\}\}_{x\in\Omega}$ is an open cover of Ω without any finite subcover.

Problem 4: Consider the metric space $\mathbb{R}$ with the usual metric.

(a) Construct an open cover of $\Omega_1 = (0, 1]$ that does not have a finite subcover.

(b) Construct an open cover of $\Omega_2 = [0, \infty)$ that does not have a finite subcover.

(c) Construct a real-valued continuous function f on Ω_1 that is not uniformly continuous. Demonstrate that for your choice of f, there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x_n, y_n \in \Omega_1$ such that $d(x_n, y_n) \leq 1/n$ and $d(f(x_n, y_n)) > \varepsilon$. Is it possible to construct such a function that is bounded? (Note: this problem was corrected by inserting a requirement that f be continuous.)

Solution:

(a) $\Omega_1 \subset \bigcup_{n=1}^{\infty} (1/(n+1),\, 1/(n-1/2))$.

(b) $\Omega_2 \subset \bigcup_{n=1}^{\infty} (n-2, n)$.

(c) Unbounded example: $f(x) = 1/x$, $\varepsilon = 0.25$, $x_n = 1/n$, $y_n = 1.5/n$.

Bounded example: $f(x) = \cos(1/x)$, $\varepsilon = 1$, $x_n = 1/(\pi 2n)$, $y_n = 1/(\pi(2n+1))$.