

Homework set 2 — APPM5440, Fall 2016

From the textbook: 1.8, 1.9, 1.10 (important), 1.12, 1.13.

In problem 1.12, it would make a good exercise to prove the statement three times, using each of the three different definitions of continuity defined in class.

Optional: 1.7.

Problem 1: Let (X, d) be a metric space, and let $\Omega \subseteq X$. Prove that:

$$\Omega \text{ is dense in } X \iff \forall x \in X, \varepsilon > 0, \exists y \in \Omega \text{ such that } x \in B_\varepsilon(y).$$

(In words: Ω is dense iff for every x in X , and for every $\varepsilon > 0$, there exists an element $y \in \Omega$ that is within distance ε of x .)

Problem 2: Suppose that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are Cauchy sequences in a metric space (X, d) . Prove that the sequence $(d(x_n, y_n))_{n=1}^\infty$ converges.

Problem 3*: The proof that every metric space has a completion that we postponed contains an important technique called the *Cantor diagonal argument*. Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration $(r^{(n)})_{n=1}^\infty$ of all real numbers in the interval $(0, 1)$. Suppose that each $r^{(n)}$ has a binary number expansion

$$r^{(n)} = 0.b_1^{(n)} b_2^{(n)} b_3^{(n)} \dots$$

(so that each $b_j^{(n)}$ is either 0 or 1) and use the “diagonal” technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor’s diagonal argument.)