Homework set 1 — APPM5440, Fall 2016

From the textbook: 1.3, 1.4, 1.5.

Problem 1: Consider the set $\mathbb{R}^n$ equipped with the norm

$$||x||_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

(a) Prove that $||\cdot||_p$ is a norm for $p = 1$.

(b) Prove that $||\cdot||_p$ is a norm for $p = 2$.

(c) Prove that $\lim_{p\to\infty} ||x||_p = \max_{1\leq j\leq n} |x_j|$.

(d*) Prove that $||\cdot||_p$ is a norm for $p \in (1,\infty)$. (See hint on next page.)

(e*) For $x, y \in \mathbb{R}^n$, let $d(x,y)$ denote the number of non-zero entries of $x - y$. Is d a metric on $\mathbb{R}^n$? Prove that $d(x,y) = \lim_{p\searrow 0} ||x-y||_p^p$. (The function $d = d(x,y)$ is often called the *Hamming distance* between x and y.)

Problem 2: Set $I = [0, 1]$ and consider the set X consisting of all continuous functions on I. Define an addition and a scalar multiplication operator that make X a normed linear space.

(a) Which of the following candidates define a norm on X:

- $||f||_{\mathrm{a}} = \sup_{0\leq x\leq 1} |f(x)|$
- $||f||_{\mathrm{b}} = \sup_{0\leq x\leq 1/2} |f(x)|$
- $||f||_{\mathrm{c}} = \sup_{0\leq x\leq 1} |f(x)|^2$
- $||f||_{\mathrm{d}} = 2 \sup_{0\leq x\leq 1} |f(x)|$
- $||f||_{\mathrm{e}} = \sup_{0\leq x\leq 1} (1+\cos x)|f(x)|$
- $||f||_{\mathrm{f}} = |f(0)| + \sup_{0\leq x\leq 1} |f(x)|$
- $||f||_{\mathrm{g}} = |f(0)|$

(b) Prove that

$$||f|| = \int_0^1 |f(x)|\,dx$$

is a norm on X.

(c) Prove that with respect to the norm given in (b), the space X is not complete.

Hint for 1d:

You may want to use the Hölder inequality: Let p and q be numbers in the interval $(1, \infty)$ such that $1/p + 1/q = 1$, and let $(\alpha_j)_{j=1}^n$ and $(\beta_j)_{j=1}^n$ be vectors in $\mathbb{R}^n$. Then

$$\sum_{j=1}^{n} |\alpha_j \beta_j| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\beta_j|^q \right)^{1/q} .$$

(You can look up a proof in, e.g., Wikipedia. You will also see that the inequality is far more general than what is stated here.)

Next let x, y be two non-zero vectors and let $r \in (1, \infty)$. Then

$$||x + y||_r^r = \sum |x_j + y_j|^r = \sum |x_j + y_j|^{r-1} |x_j + y_j| \leq \sum |x_j + y_j|^{r-1} \big(|x_j| + |y_j|\big).$$

Now use the Hölder inequality for suitably chosen p and q.