Applied Analysis (APPM 5440): Final exam
7:30pm – 10:00pm, Dec. 11, 2016. Closed books.

Name:____

Problem 1: (16p) No motivations required for these problems. 4p each.

(a) Let X be a set, and let $\mathcal{T}$ denote a topology on X. Define what it means for $\mathcal{T}$ to satisfy the Hausdorff property.

(b) Let X denote a Banach space. Mark the following statements as true/false:

	TRUE	FALSE
If $(T_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{B}(X)$ of compact operators that converges in norm to an operator T, then T is necessarily compact.		
Let $S, T \in \mathcal{B}(X)$. If S is compact, then ST is compact.		
Let $S, T \in \mathcal{B}(X)$. If S is compact, then TS is compact.		
Let $S, T \in \mathcal{B}(X)$. If S and T are both compact, then $S + T$ is compact.		
Let $S, T \in \mathcal{B}(X)$. If S is compact, then $S + T$ is compact.		

(c) Set $I = [0, 1]$ and $X = C(I)$. (We use the standard norm on X.) Define the subset

$$A = \{u \in X : \ u \text{ is continuously differentiable and } \|u'\| \leq 1\}.$$

Describe the closure $\bar{A}$ of A:

Is $\bar{A}$ a compact set (yes/no)?

(d) Set $H = L^2([-1, 1])$, and define $T \in \mathcal{B}(H)$ via $[Tu](x) = 2u(-x)$. Let $S \in \mathcal{B}(H)$ be an operator for which you know that $\|S\| \leq c$, where c is some positive number. Are there any values of c for which you can say for sure that the operator $T - S$ has closed range?

Answer:

Problem 2: (16p) Let H denote a Hilbert space. Prove that for every element $\varphi \in H^*$, there exists a unique $y \in H$ such that

$$\varphi(x) = (y, x), \qquad \forall x \in H.$$

Problem 3: (16p) Set $I = [0, \pi]$ and let H denote the Hilbert space $H = L^2(I)$ with the usual norm. Define $f, g, h \in H$ via

$$f(x) = \sin(x), \qquad g(x) = \sin(3x), \qquad h(x) = x.$$

Set $N = \text{Span}\{f, g\}$, and $M = N^\perp$. Evaluate

$$d = \inf_{u \in M} \|h - u\|.$$

In the event that you make any computational errors, your score on this problem will depend strongly on whether you clearly describe the process you use to determine d.

Problem 4: (16p) Set $I = [0, 2]$, set $X = C(I)$, and let k be a continuous function on $I \times I$. Consider the operator $T \in \mathcal{B}(X)$ defined by

$$[Tu](x) = \int_0^2 k(x, y)\, u(y)\, dy, \qquad x \in I.$$

(a) State the Arzelá-Ascoli theorem.

(b) Prove that the operator T is compact.

Problem 5: (16p) Let X denote the space of all continuous functions on $\mathbb{R}$ that are periodic with period 1. In other words, if $u \in X$, then

$$u(x) = u(x + 1), \qquad \forall x \in \mathbb{R}.$$

We equip X with the norm

$$\|u\| = \sup_{x \in [0,1]} |u(x)|.$$

Observe that a function u in X is uniquely defined by its values on the interval $I = [0, 1]$ (or on $[0, 1)$, for that matter, since $u(0) = u(1)$). Define for $n = 1, 2, 3, \dots$ the operators

$$[T_n u](x) = u(x - 1/n).$$

(a) (6p) Does $(T_n)_{n=1}^\infty$ converge strongly? Please motivate your answer carefully.

(b) (6p) Does $(T_n)_{n=1}^\infty$ converge in norm? Please motivate your answer carefully.

(c) (4p) Do your answers change if X is instead equipped with the norm $\|u\| = \int_0^1 |u(x)|\, dx$?